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THE RESTRICTED PROBLEM OF THREE BODIES

BY

JAMES H. BARTLETT

København 1964
Kommissionær: Ejnar Munksgaard

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Synopsis

The case of two equal finite masses has been studied extensively, using Thiele[9] variables, a modified Runge-Kutta method, and an electronic computer. The main classes of Strömgren[4] have been traced continuously from beginning to end, and seven new classes are reported. For convenience, a class is represented by an eigensurface in (E, F, K) space, where K is the Jacobi integral. General methods of locating periodic solutions, in particular asymmetric ones, are discussed. Initial and final conditions, in the form of tables and curves, are given for more than 800 periodic orbits.

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The restricted problem of three bodies, which consists in the determination of the motion of a body of infinitesimal mass under the gravitational action of two other bodies of finite mass, has been investigated by many theoreticians and computers. POINCARÉ (1) and BIRKHOFF (2) obtained valuable general results; DARWIN (3), STRÖMGREN and his school (4), SHEARING (5), and GOUDAS (6) have made extensive computations, have shown in detail how some of the periodic solutions look, and have made limited studies of their stability.

A comprehensive picture of the simpler periodic solutions is perhaps best attained by means of the concept of a *class* of periodic orbits, which concept was introduced and employed effectively by STRÖMGREN for the case of equal finite masses. If one periodic orbit is known, then one may vary either the first integral (Jacobi constant) or the mass ratio, or both, and then adjust the initial conditions continuously to give another periodic orbit; the family of orbits so obtained is said to constitute a class. A good understanding should then be reached if one can describe and explain the internal structure of the simpler classes, and to show how and why various classes are interrelated. It is in this sense that the present program has been undertaken, rather than for the purpose of calculating orbits with extreme exactness.

In the present article, which is restricted to the case of equal finite masses, there are presented curves and tables which show how most of the main classes of STRÖMGREN develop continuously. (STRÖMGREN was, in most cases, only able to give a few typical members of the class, because his work was done before the advent of the modern electronic computer). Class (g), which was started by BURRAU and STRÖMGREN (7), and carried through about one-half of its development by the late P. PEDERSEN, is given completely. Seven new classes: (λ), (μ), (ν), (α), (β), (γ), and (δ), the the latter three of which are just as important as class (g), are reported for the first time.* We also indicate how to determine several classes of asymmetric orbits, and we show how one may use a suitable mapping to find all the periodic solutions. Inspection reveals that continuation to the case of unequal masses is straightforward, and work is proceeding along these lines. Definite statements about ultimate stability are, however, very difficult to make, because they require a very precise knowledge

* Presumably our (β) class is the same figure-of-eight class $(C'; C'')$ predicted by Darwin, for the case $m_1 = 10\,m_2$, in part V of his second paper (3).

of the mapping near a fixed point (elliptic). MOSER [8] has shown that stability can occur for certain special mappings, but it remains to be seen whether our mappings fall into this category.

Equations of Motion

Suppose we have two bodies S and J with masses m_1 and m_2 respectively, which execute circular motions about their common center of gravity, and that the distance SJ between them has magnitude 2 units. Let us study the motion of a third body P which has vanishingly small mass and moves in the same plane as S and J do.

Let there be a coordinate system (x, y) fixed in the plane, with origin O at the center of gravity. Set $SO = r_1$, $OJ = r_2$, $SP = r$, $PJ = \varrho$. P, S, and J have as coordinates (x, y), (x_1, y_1) and (x_2, y_2). The equations of motion for P are

$$\left.\begin{aligned} \ddot{x} &= k^2\, m_1\, (x_1 - x)/r^3 + k^2\, m_2\, (x_2 - x)/\varrho^3 \\ \ddot{y} &= k^2\, m_1\, (y_1 - y)/r^3 + k^2\, m_2\, (y_2 - y)/\varrho^3 \end{aligned}\right\} \quad (1)$$

Now let us refer the motion to a rotating coordinate system (ξ, η), where the ξ-axis lies along SJ. The angular velocity is

$$\omega = (k/2)\left[(m_1 + m_2)/2\right]^{1/2} = k\,(M/8)^{1/2}$$

The equations of motion in this system are

$$\left.\begin{aligned} &\ddot{\xi} - 2\,\omega\,\dot{\eta} - \omega^2\,\xi + k^2 m_1 (\xi + r_1)/r^3 + k^2 m_2\,(\xi - r_2)/\varrho^3 = 0 \\ &\ddot{\eta} + 2\,\omega\,\dot{\xi} - \omega^2\,\eta + k^2 m_1\,\eta/r^3 + k^2 m_2\,\eta/\varrho^3 = 0\,. \end{aligned}\right\} \quad (2)$$

If we set $\omega = 1$, then $k^2 M = 8$, and equations (2) may be written as

$$\ddot{\xi} - 2\,\dot{\eta} = \partial\, U/\partial\,\xi\,,\ \ddot{\eta} + 2\,\dot{\xi} = \partial\, U/\partial\,\eta \qquad (3)$$

where

$$2\,U = \xi^2 + \eta^2 + 8\,(1 + \gamma)/r + 8\,(1 - \gamma)/\varrho \qquad (4)$$

with

$$\gamma = (m_1 - m_2)/(m_1 + m_2),\ \text{and}\ (m_2/M) = \frac{1}{2}(1 - \gamma)\,.$$

Equations (3) have the first integral

$$\dot{\xi}^2 + \dot{\eta}^2 = 2\,U - K \qquad (5)$$

where K is the Jacobi constant (Strömgren's notation).

Equations (2) are singular at $r = 0$ and $\varrho = 0$, so that one cannot treat collision orbits on a par with other orbits. To overcome this difficulty, THIELE [9] introduced a transformation which allows one to vary parameters of a family (class) of orbits smoothly, paying no special heed to collision orbits. This transformation is

$$\left.\begin{aligned} \xi &= \cosh F \cos E + \gamma \\ \eta &= -\sinh F \sin E \\ d\psi &= (\omega/r\varrho)\, dt \end{aligned}\right\} \quad (6)$$

We have the further equations

$$r_1 = 1-\gamma, \; r_2 = 1+\gamma$$

$$r = \cosh F + \cos E, \; \varrho = \cosh F - \cos E$$

and

$$r\varrho = (1/2)\,(\cosh 2F - \cos 2E).$$

When $F = 0$, then $\xi = \cos E + \gamma$, $\eta = 0$, which corresponds to the ξ-axis between the 2 masses. When $E = 0$, $\xi = \cosh F - 1 + r_2$, $\eta = 0$, which represents the ξ-axis between m_2 and $+\infty$. Similarly, when $E = \pm\pi$, $\xi = 1 - \cosh F - r_1$, $\eta = 0$, describing the ξ-axis between m_1 and $-\infty$. The line $E = \pi/2$ has coordinates $\xi = \gamma$, $\eta = -\sinh F$, and is the locus of points equidistant from the 2 masses.

In what follows, a dot will denote differentiation re ψ rather than re t.

The equation for the first integral, namely (5), now transforms into

$$\left.\begin{aligned} \dot{E}^2 + \dot{F}^2 &= (1/8)\,(\cosh 4F - \cos 4E) - (T/2)\,(\cosh 2F - \cos 2E) + 16 \cosh F \\ &\quad + (\gamma/2)\,(\cos E \cosh 3F - \cos 3E \cosh F - 32 \cos E) = 2H \end{aligned}\right\} \quad (7)$$

where $T = K - \gamma^2$.

The differential equations (3) themselves become

$$\left.\begin{aligned} \ddot{E} &= (\cosh 2F - \cos 2E)\,\dot{F} + \partial H/\partial E \\ \ddot{F} &= -(\cosh 2F - \cos 2E)\,\dot{E} + \partial H/\partial F. \end{aligned}\right\} \quad (8)$$

In the case of equal masses, $\gamma = 0$, $T = K$, and

$$\left.\begin{aligned} \ddot{E} &= (\cosh 2F - \cos 2E)\,\dot{F} + (1/4)\sin 4E - (T/2)\sin 2E \\ \ddot{F} &= -(\cosh 2F - \cos 2E)\,\dot{E} + (1/4)\sinh 4F - (T/2)\sinh 2F + 8\sinh F \end{aligned}\right\} \quad (9)$$

When $\gamma \neq 0$, the right hand sides of (9) have additional terms

$$\left.\begin{aligned} \Delta\ddot{E} &= -(\gamma/4)\,(\sin E \cosh 3F - 3\sin 3E \cosh F - 32 \sin E) \\ \Delta\ddot{F} &= -(\gamma/4)\,(-3\cos E \sinh 3F + \cos 3E \sinh F). \end{aligned}\right\} \quad (10)$$

The present article will deal with equal masses ($\gamma = 0$), and so equations (7) and (9) will be applicable. Let us first make some general remarks about the invariance properties of equation (9).

This equation is still the same if E is replaced by $E + \pi$, which amounts, if $\gamma = 0$, to replacing ξ by $-\xi$ and η by $-\eta$. In other words, for equal masses the physical system remains invariant under a rotation of $180°$.

The reversed motion is obtained either by replacing ψ with $-\psi$ or by changing F to $-F$. The latter is equivalent to the transformation $\xi' = \xi$, $\eta' = -\eta$, or reflection about the ξ-axis, and also, from the preceding, to reflection about the η-axis, i.e. $\xi' = -\xi$, $\eta' = \eta$. The equations (9) are thus invariant under $E' = \pi - E$, $F' = -F$.

Periodic Solutions

The periodic solutions occupy an important place in the theory of the equations, because some of them are quite stable and the system stays together for a long time. Therefore our first task will be to locate where the periodic solutions are, in general. We need only determine the simpler periodic solutions, because Birkhoff has proved the existence of solutions which have periods that are multiples of the basic period. A revision of this proof of Birkhoff's Fixed Point Theorem has been given by Siegel(10).

If the motion of a dynamical system is to be periodic, this means that after a period τ the dynamical variables return to their original values. Alternatively stated, one must in general solve a system of non-linear ordinary differential equations, subject to the boundary conditions that the final positions and velocities must have the same values as the initial ones.

Let us first consider motion in one dimension subject to a force which is not explicitly dependent on the time. The equation of motion $\ddot{u} + f(u) = 0$ has a first integral $\frac{1}{2}\dot{u}^2 = h - V(u)$, where h is constant and V is the potential energy. If V has a minimum, then librations will occur in the valley of V, and the period may be determined by a quadrature. Given h and u, one can determine the velocity $\dot{u}$ except for sign. The periodic motions may be easily visualized by drawing the trajectories in the phase plane $(u, \dot{u})$.

It is somewhat more difficult but still feasible to characterize periodic motion in 2 dimensions, such as is the case for the restricted 3-body problem. Let the Jacobi constant T have a definite value, and consider the totality of periodic motions belonging to this value. They will be closed curves in the (E, F) plane, which we shall call *eigencurves*. These curves can be symmetric with respect to (1) the ξ-axis (2) the η-axis (3) both the ξ-axis and the η-axis or (4) neither the ξ-axis nor the η-axis.

[The symmetry properties of the equations do not by any means exclude asymmetric solutions, but such solutions have hitherto been largely ignored because they are somewhat more complicated and also less easy to locate. However, Strömgren (4) (see Tableau V, fig. 8) gives one example and Rabe (11) some others]. When T is varied continuously, the eigencurves also change continuously, and generate *eigensurfaces* in (E, F, T) space. The totality of these surfaces is thus a representation of periodic motion for our problem. More generally, one can let γ, the mass-ratio parameter, vary and see how the eigensurfaces change. Each distinct surface is said to represent a *class* of periodic solutions.

Location of Periodic Solutions

For a given T, assume that the eigencurve is cut by some line such as E = const. Then we may take this value as our initial and final value of E, and consider the transformation S which carries the initial value of E over into its final value. This transformation will simultaneously take the initial values of F and $\dot{E}$ into new ones, i.e. $S(F_i, \dot{E}_i) = (F_f, \dot{E}_f)$. This may be regarded as mapping the $(F, \dot{E})$ plane into itself. Now, for any a, the equation $S(a, \dot{E}_i) = (a, \dot{E}_j)$ gives the intersection of the map of $F = a$ with the line $F = a$ itself, and if a be allowed to vary, we obtain the locus of "a" intersections. Likewise, from $S(F_i, b) = (F_j, b)$, we obtain the locus of "b" intersections, and the intersections of the two loci will be the fixed points $S(\alpha, \beta) = (\alpha, \beta)$. These fixed points characterize the periodic solutions which intersect the given line E = const.

Such a method of obtaining periodic solutions for a given T is systematic and thorough, and it guarantees that important periodic solutions will not be overlooked. In practice, it can involve excessive labor of computation, so that it is often better to use other methods based upon the physical nature of the problem. These will now be discussed.

The simplest type of periodic solution is that where the particle is at rest in the rotating system, which may occur at the libration points. These are five in number, namely $L_1: E = \pm\pi/2,\ F = 0$; $L_2: E = 0,\ F = \pm 1.5206$; $L_3: E = \pi,\ F = \pm 1.5206$; L_4: $E = \mp\pi/2,\ F = \pm 1.316958$; and $L_5: E = \pm\pi/2,\ F = \pm 1.316958$. Also, one might expect relatively simple periodic solutions near one of the masses, since the influence of the other mass would be relatively small there.

According to Strömgren, each class has a natural beginning and a natural end, and these can coincide. Furthermore, the beginning and end will, if not infinite, be related to the positions of the masses or of the libration points. (This will be made more explicit below). This principle enables one to discover at least one periodic orbit belonging to a class. It is then a simple matter to vary an appropriate initial condition, either E or F, and to determine how T must vary to preserve periodicity. Thus the whole class may be traced out.

Strömgren confined his attention mainly to orbits which were symmetrical either with respect to the E-axis, or to the F-axis, or perhaps both. This makes the location of a periodic orbit rather easy for a given T, because one knows that the initial inclination is perpendicular to one of these axes. Then it is only necessary to vary the distance along the axis until the final boundary conditions are fulfilled, provided of course that a periodic orbit of the desired type does exist for the value of T in question.

For many classes, a natural termination is an asymptotic orbit spiraling out from L_4 (or L_5). This can be symmetric with respect to the ξ-axis and spiral into L_5, or symmetric with respect to the η-axis and spiral into L_4, or completely asymmetric and spiral into L_4. Examples of all these orbits are given by Strömgren[(4)]

(see Fig. 15 and Tableau V, loc. cit). If one knows these limiting orbits, then a technique must be devised for finding the other members of the class. One feasible method is to make use of the coiling property of the eigensurface, which will now be explained and then applied.

Class (k) is a rather simple one and is well-suited for the demonstration of the coiling. If one considers the profile of the eigensurface corresponding to the plane $E = 0$, this is a curve which begins with a spiral about a point $F = F_i$, $K = 11$ and ends with a spiral about another point $F = F_j$, $K = 11$. The points F_i and F_j are the F-intercepts of Strömgren's asymptotic orbits I and II. Since this coiling does occur for all classes (for $\gamma = 0$) which terminate in asymptotic orbits, we are assured that there will be an orbit of the class at a finite distance from F_i (or F_j) and with $K = 11$. This simple observation enables us to find such an orbit readily, provided one knows what the limiting orbit of the class is. And it does not matter greatly if the orbit is an asymmetric one.

Some limiting orbits for asymmetric classes can be found in a simple and systematic manner. These are the orbits which begin and end at L_4. An orbit which ends at L_4 will be the mirror image in the η-axis of one which begins at L_4. Now all the asymptotic orbits calculated by STRÖMGREN cross the η-axis at points with ordinates η_o, not far from L_4, and at an angle of $57°42.25'$ with the η-axis. If we plot the slope $d\eta/d\xi$ at a subsequent crossing versus η_c, the value of η at the crossing, the resulting curve C may be regarded as consisting of two parts, C^+ if $d\eta/d\xi > 0$ and C^- if $d\eta/d\xi < 0$. Now replace all ordinates of C^- by their negatives, thereby obtaining the mirror image $\overline{(C^-)}$, and intersect this image with C^+. The intersections will give us curves which begin and end at L_4, and these need not be symmetric either to the ξ-axis or to the η-axis. If not, we have a limiting orbit for an asymmetric class.

Structure of the Main Classes

In order to demonstrate how the principal simple symmetric classes develop, we show two profiles of the eigensurfaces, together with enlarged drawings where necessary. Figure 1a is a plot of K vs. E_i for $F_i = 0$ and $\dot{F}_i > 0$, Figure 1b an enlargement of part of Figure 1a. Figure 2a is a plot of K vs. F_i for $E_i = 0$ and $\dot{E}_i > 0$, and Figures 2b and 2c are corresponding enlargements. (The restrictions $\dot{F}_i > 0$ and $\dot{E}_i > 0$ are introduced for purposes of clarity in representation).

If two profiles intersect, the point of intersection corresponds to an orbit which is common to the two classes. For instance, in Figure 1a, the initial values E_i are then the same, $F_i = 0$, K has the same value, and $\dot{F}_i$ is positive in both cases. The orbit is uniquely determined by these initial conditions and the differential equations, and so must be a common one. However, if $\dot{F}_i$ had been negative in one case and positive in the other, the conclusion would not be correct, and this is the reason for our convention that $\dot{F}_i$ shall be positive for the profile.

If two profiles are close to each other, the orbits of the two classes will be close initially, and usually close over an appreciable interval of time. Eventually they will diverge because, belonging to different classes, they will satisfy different final boundary conditions, in general. However, since both orbits are periodic, this divergence is later compensated for by a corresponding convergence, so that no immediate conclusion about stability can be drawn.

In Figure 1a, an intersection of a profile with the K-axis marks the point $E_i = 0$, $F_i = 0$, so that we have a "periodic" ejection orbit with $\dot{E}_i = 0$. For Figure 2a, similar intersections give "periodic" ejection orbits with $\dot{F}_i = 0$. (The word "periodic" is used here in a loose sense. These orbits are not actually periodic physically, but the nearby orbits of the class are, and there is a perfectly smooth transition through the ejection orbit). If a class does have an ejection orbit, it becomes very easy to locate the class. Accordingly, we show in Figures 3a and 3b ejection orbits for $\dot{E}_i = 0$, in Figures 4a and 4b ejection orbits for $\dot{F}_i = 0$, and in Figures 5a, 5b, and 5c ejection orbits for $K = 10$ as a function of angle of ejection.

According to Strömgren, a class is either closed or has a natural beginning and a natural end. In practice, this means that the eigensurface is either closed, becomes infinite at one of the masses, stretches to infinity, or is bounded by a limiting curve (asymptotic orbit). Combinations of these latter possibilities occur, as the general rule.

A class is a continuous family of periodic orbits with certain symmetry properties (including complete asymmetry). In some cases, one may integrate over one half period or even one quarter period, specifying certain initial and final boundary conditions. These conditions remain the same throughout the class, while other properties, such as whether the motion is retrograde or direct, or simply- or multiply- periodic, may not. Strömgren did not confine himself to classes with simply-periodic orbits throughout, and he did omit classes which are as simple as the ones he included.

Three fairly simple "open" classes are (c), (f), and (m). Class (c) is defined by $F_i = 0$, $\dot{E}_i = 0$, $E_f = \pi/2$, and $\dot{F}_f = 0$. It starts with the libration point L_1, $K = 16$, $E_i = \pi/2$. The orbits in the neighborhood are simply-periodic and retrograde, and all are symmetric with respect to $E = \pi/2$ and $F = 0$. Class (f) is defined by $F_i = 0$, $E_i < 0$, $\dot{E}_i = 0$, $\dot{F}_i > 0$, $E_f = 0$, $\dot{F}_f = 0$. It starts at mass m_2, $K = \infty$, $E_i = 0$. The nearby orbits are simply-periodic and retrograde, and all are symmetric with respect to $E = 0$ and $F = 0$. For both classes, K falls rapidly at the start and goes through a series of damped oscillations as E_i decreases (or as F_f increases). The (c) profile stays below the (f) profile, running more or less parallel to it; the two profiles cannot intersect because of the different symmetries. The theory of this behavior at large distances has been given by J. P. Møller [12].

The class (m) has been included in the tables for completeness, but not in the figures (because the maximum value of K is only about $K = -2.47$. This class of retrograde periodic orbits around the two finite masses also has retrograde motion in the fixed coordinate frame (due to the high velocities at all points of the orbits).

The class begins with circles of infinite radius but zero period (in the limit, of course, as K goes to $-\infty$ and F_i goes to $+\infty$). As the class develops by closing in on the masses, the orbits become ellipses of increasing eccentricity. In the limit (as K again goes to $-\infty$, but F_i now goes to zero) the orbits become rectilinear orbits between the two masses, with zero period (i.e. an ellipse of eccentricity one).

Class (n) is defined to be symmetric with respect to just the E-axis, and retrograde. Its profile for $F_i = 0$ is periodic in E, with period π, as are the differential equations themselves for $\mu = 1/2$ ($\gamma = 0$). This class is therefore closed in the (ξ, η) system, the one with immediate physical meaning. Between the minimum value of K and the value at $E_i = 0$ the orbits are doubly-periodic about one of the masses in the (ξ, η) system, but from the collision value of K to its maximum they are simply-periodic librations between the masses. At the minimum value of K there is a common orbit with class (f), since there is then symmetry also with respect to the F-axis. At the maximum value of K there is an orbit in common with class (c), and in this case symmetry with respect to the η-axis ($E = \pi/2$).

Class (a) is defined to be symmetric about the F-axis only, and retrograde. Its simplest member is the stationary libration point L_2. The class is a closed one, so has no beginning or end, but it is convenient to start with L_2 and follow the development. Here the value of K is a maximum, corresponding to the fact that the velocity is zero. But this value falls rapidly as F_i is increased. The orbits at first are simply-periodic librations about L_2, and remain so until the ejection orbit is reached. The value of K goes through a minimum somewhat before this. After the ejection orbit, the motion is doubly-periodic about the mass m_2 in the (ξ, η) system. The value of K increases from that at ejection to a maximum and then drops to another minimum, after which the orbits are retraced in reverse sequence to L_2. At this minimum for K, there is an orbit in common with class (f), so that for this point only there is symmetry about both the E-axis and the F-axis. (The above development can be followed by referring to Figure 2a, and observing that the full line is a semi-profile of the eigen-surface. Each orbit has two intercepts on the F-axis, of the same sign for librations and of opposite sign for motion around the mass). From Figure 2a, it is evident that there is a maximum value of K just after ejection and that the maximum value of F_i occurs soon after this, i.e. is not coincident with ejection. It should be noted that the data of STRÖMGREN were extremely scanty for class (a), so that it was impossible to construct an accurate profile, such as is here presented.

As a third and final example of a closed class, let us consider class (δ), the profile of which is shown in Figures 2a and 2c. This class, which was unknown to STRÖMGREN is about as simple as class (a). It is a class which is generally symmetric only about the F-axis, and is partly retrograde and partly direct. At its maximum K it has an orbit in common with class (g), as also at its minimum K. The first is direct and the second retrograde, and both are symmetric with respect to both the E- and F-axes. At the maximum K, the motion is doubly-periodic, and direct about the mass. It remains so until the ejection orbit is reached, when it changes to a direct, simply-periodic libration. This behavior persists until an intermediate minimum for K, where the orbit has but one intercept on the F-axis. The libration then changes to a retrograde one until the second ejection orbit, when the motion becomes doubly-periodic and retrograde around the mass. After K reaches the second minimum, where the orbit is in common with the (g) class, the development of the class proceeds in reverse sequence back to the maximum value for K. The general behavior of class (δ) is in some respects similar to that of class (a), even though the latter has only retrograde motions.

The remaining classes in this paper are all ones which terminate on asymptotic-periodic orbits from L_4 and L_5. We have completed class (g), made class (k) more precise and complete, and have discovered and traced out 6 other new classes, which we designate by (α), (β), (γ), (λ), (μ), and (ν). Class (α) was the first to be discovered, and is shown on Figures 2a and 2b and also tabulated. It is complicated and of minor importance, so that it will only be defined, as consisting of orbits which start out with $E = 0$, $\dot{F} = 0$, $\dot{E} > 0$ and after exactly one-half period satisfy these same conditions (but F has assumed a value different from the initial one). The other classes in this group are only a small fraction of the possible ones associated with L_4 and L_5, as is easily seen from the systematic method of generation. STRÖMGREN listed 5 simple periodic-asymptotic orbits and mentioned that one could combine these half-orbits together. As a matter of fact, any such combination can be regarded as the limiting orbit of a class, and we may trace out the class by the method described previously. The classes presented in this paper are simple ones which have been easy to locate, and it is hoped that their study will furnish a good picture of the general behavior.

Figure 1b shows the profile of class (g) in the neighborhood of our 2 new classes (β) and (γ), together with the profiles of the latter two classes. Class (β) consists of trajectories which are symmetric with respect to both the E- and F-axes. One quarter of each trajectory is a curve which starts up normal from the E-axis and ends normal to the F-axis and with $\dot{E} > 0$. Let us denote by VII that periodic-asymptotic orbit which proceeds from L_4, crosses the F-axis once and then strikes the F-axis normally at about $F = 1.75$. Then one-half of one limiting orbit for the (β)-class is composed of half-orbits III and VII, while the other half-orbit consists of IV and VII. The motion is for most of the class direct around the mass, but there is a small portion where $E_i < 0$ and the motion is retrograde. Class (γ) is symmetric in general only about the E-axis, as is class (n), but it is, for the most part, direct. For each permissible value for K, there is at least one pair of intercepts on the E-axis. When K is maximum, one intercept is the negative of the other. The two limiting orbits are each composed of 2 half-orbits (combined in opposite ways at the two spiral ends), one of which is Strömgren's III, while the other (XII) comes from $\eta_i = 1.719051$ and intercepts the E-axis normally at -0.037282.

The interrelations of the above classes are interesting. Class (β) has as a quarter-trajectory a curve ending with $\dot{E} > 0$, while class (g) has for its similar curve one ending with $\dot{E} < 0$, so that the profiles cannot possibly intersect. Class (γ) has a trajectory symmetric about the F-axis at its maximum K, so that class (g) can and does intersect class (γ) there. The (g)-profile runs very close to the (β)-profile over a considerable range, indicating that only a small change in slope near the end of the quarter-trajectory will cause a change from class (g) to class (β). Class (β) can perhaps be regarded as a sort of combination of class (g) with class (a), and as corresponding to Darwin's figure-of-eight class (C', C'') for $\mu = 10/11$ ($\gamma = -9/11$). In Figure 2b, we see that its F-profile is between those for (a) and (g).

Figure 2b shows partial profiles of classes (α), (g), (ν), (β), and (a). All these classes except (a) have limiting orbits consisting in part of semi-orbit VII. If the profiles were drawn to completion, they would all spiral around the limiting F-intercept of VII at $K = 11$. This happens for one end of (α), one end of (g), and both ends of (β) and (ν). Class (ν) starts out normal to the F-axis with $\dot{E} > 0$ and strikes the η-axis ($E = -\pi/2$) normally. Since these boundary conditions are not compatible with those for (g) and (β), the profiles for these two classes will not be intersected by that for class (ν). Class (ν) has a maximum at about $K = 13.72$ near L_2, and a minimum at $K = 9.08$. At one spiral end it has as limiting half-orbit the Limiting Orbit VII. The other end has a double-spiral combination of Limiting Orbit VII plus half of another orbit due to Strömgren (loc. cit., Tableau V, Figure 2) as its half-orbit. As the limiting orbits of (α), (g), (ν), and (β) are approached, the profiles come very close together, which reflects the fact that the limiting orbits all have semi-orbit VII in common.

Figure 2c shows in detail the relation between classes (g), (δ), (α), (k), and (μ). For $F_i > 0$, classes (g), (δ), (α), and (k) run close together, because there is a large loop in the trajectory and only a small variation in F_i is necessary to change the orientation of this loop, and hence to change from the satisfaction of one final boundary condition (say $\dot{F} = 0$ at $E = \pi/2$) to another (say $\dot{E} = 0$ at $F = 0$). Class (k), which is symmetric about the F- and η-axes, runs between the 2 limiting half-orbits I and II, with large variation in K. Class (μ), which is just symmetric about the F-axis, runs between a limiting orbit I + V at one end and II + V at the other, with only small variation in K. This may be due to the fact that orbit V corresponds to a large F_i, and that only a small change in the velocity there can lead to a large change of slope at the other end of the trajectory (at small F_i). The orbits of (μ) loop around both L_4 and L_5, but remain outside the mass m_2 for most of the class (i.e. except for that part which approaches Limiting Orbit II, with $F_i > 0$). They differ from the orbits of the (k) class because of the outer portion associated with Limiting Orbit V.

Figure 2a shows class (l) and how it spirals around the F_i of V. Another new class, (λ), spirals about a point with F_i slightly less than that for V, and corresponding to an asymptotic orbit (XIII) starting from L_5, looping near L_4, and then hitting the F-axis normally. The K for this class falls off with increasing F_i. One orbit of this class was erroneously assigned by Strömgren to class (l).

To give a general idea of how the trajectories vary with F_i and E_i, we have chosen a convenient value of $K = 12.5$ and plotted the corresponding trajectories.

Figure 6a covers the range from $F_i = -1.25$ to $F_i = -0.5$, Figure 6b shows $F_i = -0.7$ to $F_i = 1.75$, and Figures 6c and 6d show how the trajectories depend on E_i (from $E_i = -1.8$ to $E_i = 0$, and $E_i = 0$ to $E_i = 1.6$). By referring to these diagrams, it is easy to see when the various boundary conditions, for periodic orbits, will be satisfied.

Figure 7 shows the different periodic orbits themselves for $K = 12.5$. It is particularly interesting to note how, when the profiles are close, the orbits themselves are close over a good portion of their paths. For example, the (δ) class is a sort of combination class, which is possible because the initial E for one (g) class orbit (with $E_i \cong 1.35$ and $F_f = -0.15$) is not too different from final E for another (g) class orbit (with $F_i \cong -0.8$ and $E_f \cong 1.2$). All that is needed is a small change of slope from the vertical to effect the transition to the (δ) class orbit. If we look at Figure 1a, the two "branches" of the (g) class run parallel and not too far apart in this region, for a wide variation in K.

Figures 8a, 8b, and 8c show, for the first time, the detailed and complete development of class (g). STRÖMGREN wrote "It is certain that in one way or another this class is "associated" with the points L_4 and L_5". But this statement lacked effective content, since the manner of association was completely unknown. P. PEDERSEN, in unpublished calculations made just before his death in 1958, followed class (g) to $K = 7.6$. We obtained more points on the profiles to this stage, and traced the class to its limiting orbit. The interpretation of its behavior is rather impossible unless one knows of the existence of classes (δ), (γ), (β), and (ν), which were discovered by us.

Class (g) begins with direct motion around mass m_2. The amplitude in both E and F directions increases until a cusp develops, followed by a loop. Now the F-profile becomes close to that of the (δ) class, with which there are 2 intersections, one at maximum K and the other at minimum K. The E-profile is at the same time close to that for class (c), as is easily possible when the loop is tight, and small changes in slope are all that are necessary to meet a desired boundary condition. Since there can be an intersection of the E-profiles of classes (g) and (γ), this will have to occur at the maximum K for class (γ), and so the (g) profile swings up to do this. Actually, the (g) profile can follow that of the (β) class for a longer stretch of development than for class (γ), and does so, as is seen in Figure 1b. However, the lower intercept of (β)-trajectories with the F-axis (where the slope is not horizontal) must remain positive. At the same time, the value of F_f for class (g) can vary smoothly and change sign. This is what happens, and at the F-ejection orbit (about $K = 10.2$) the two E-profiles part company, that for the (g) class suffering a sharp reversal of direction. The value of E_i now becomes negative, the motion retrograde, and the E- and F-amplitudes increase steadily. The loop has disappeared and the motion is now triply-periodic, although the class does have a common orbit with the simply-periodic retrograde class (f), at about the minimum K for class (g). From here on, the general development is that of slow changes in E and F, and rapid ones in K. The middle part of the quarter trajectory, which is at first in the fourth quadrant ($E>0$, $F<0$), moves to the left and up, across a skew-angle ejection orbit ($E = 0$, $F = 0$), and then develops first a cusp and then a loop. The final (limiting) half-orbit is a double spiral around L_4 consisting of asymptotic orbits VII and VIII. The first neighboring class to be reached is (ν), with trajectories normal to $E = -\pi/2$. Then the trajectories become close to those of class (β) and remain so half-way around L_4, because this part of the limiting orbit (VII) is common. Since class (ν) has a maximum $K = 13.72$, and since class (g) cannot intersect it, the profile of class (g) must go around that of class (ν), which it does, to become asymptotic to the class (β) profile. This accounts completely for the F-profile of class (g), and the remainder of the E-profile does not seem to offer anything of interest.

Figure 9 shows the development of the closed (δ) class, which is rather complicated, from maximum K to minimum K. The development from minimum K to maximum K is obtained by taking the mirror images (about the E-axis) of the orbits shown in the figure.

Finally, Figure 10 shows the periodic-asymptotic orbits of Strömgren, as well as nine others which we have found. These latter are necessary for an understanding of how various classes, in particular (g), terminate.

In order that the work be truly quantitative, it is imperative to give initial conditions for the periodic solutions which we have obtained. With these and an electronic computer, one can reproduce any orbit desired. In our calculations, the main work has consisted in varying the initial conditions so that the final ones would be satisfied, and the actual solutions (orbits) have been printed out only for cases which seemed particularly interesting, such as the (g) class. Our initial and final conditions, as well as the elapsed "time" $x = \Delta\psi$, are given in the tables. Also, the last table gives the initial and final conditions for all the asymptotic-periodic half-orbits shown in Figure 10.

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Class (a)

Initial Conditions: $E_i = 0$; $\dot{F}_i = 0$; $\dot{E}_i > 0$.

Final Conditions: $E_f = 0$; $\dot{F}_f = 0$; $\dot{E}_f < 0$.

Note: To obtain the remainder of the class take the mirror images, i.e. the values

$$F_i' = -F_f;\ F_f' = -F_i$$

K	F_i	F_f	x
13.81871	−1.500491	−1.540000	0.4985
13.80732	−1.489491	−1.550000	0.4989
13.79078	−1.478037	−1.560000	0.4994
13.76877	−1.466099	−1.570000	0.5001
13.74096	−1.453646	−1.580000	0.5009
13.70697	−1.440643	−1.590000	0.5020
13.66642	−1.427047	−1.600000	0.5032
13.64355	−1.420013	−1.605000	0.5040
13.61888	−1.412817	−1.610000	0.5047
13.57192	−1.400000	−1.618617	0.5062
13.40306	−1.360000	−1.643399	0.5117
13.20556	−1.320000	−1.665695	0.5180
13.09854	−1.300000	−1.675304	0.5219
12.98614	−1.280000	−1.684904	0.5258
12.50000	−1.200282	−1.718210	0.5433
12.49823	−1.200000	−1.718287	0.5434
11.97114	−1.120000	−1.744926	0.5645
11.49330	−1.050000	−1.765665	0.5849
11.00000	−0.977948	−1.783724	0.6084
10.47530	−0.900000	−1.800846	0.6364
9.829703	−0.800000	−1.821464	0.6761
9.234006	−0.700000	−1.841319	0.7206
8.706927	−0.600000	−1.862726	0.7703
8.274730	−0.500000	−1.887448	0.8259
7.976367	−0.400000	−1.917682	0.8889
7.873906	−0.300000	−1.954912	0.9616
8.048263	−0.200000	−1.998023	1.0467
8.512360	−0.100000	−2.039212	1.1426
8.929865	−0.020000	−2.061301	1.2145
9.009506	0	−2.065817	1.2291
9.079741	0.020000	−2.068127	1.2427
9.152730	0.050000	−2.072029	1.2592
9.200000	0.126497	−2.074709	1.2861
9.000000	0.231699	−2.076561	1.2938
8.500000	0.368000	−2.069787	1.2819
8.000000	0.478326	−2.053335	1.2662
7.398739	0.600000	−2.030726	1.2459
6.896330	0.700000	−2.005726	1.2291
6.404044	0.800000	−1.975115	1.2127
5.935352	0.900000	−1.938862	1.1970
5.502325	1.000000	−1.896388	1.1823
5.115463	1.100000	−1.847226	1.1689
4.784783	1.200000	−1.790620	1.1572
4.335157	1.400000	−1.650040	1.1407
4.243615	1.500000	−1.562275	1.1373

Class (c)

Initial Conditions: $F_i = 0$; $\dot{E}_i = 0$; $\dot{F}_i > 0$.

Final Conditions: $E_f = \pi/2$; $F_f = 0$; $\dot{E}_f > 0$.

K	E_i	F_f	x
15.85000	1.538030	0.142749	0.5453
15.81000	1.533797	0.16084	0.5455
15.79000	1.531834	0.169227	0.5456
15.77051	1.530000	0.177017	0.5457
15.35576	1.500000	0.300644	0.5473
14.41663	1.450000	0.487065	0.5511
13.41040	1.400000	0.647834	0.5554
12.50000	1.350854	0.784318	0.5597
11.69009	1.300000	0.907716	0.5640
11.00000	1.247388	1.020574	0.5686
10.49915	1.200000	1.112129	0.5729
9.776064	1.100000	1.281767	0.5831
9.456517	1.000000	1.427175	0.5967
9.505903	0.900000	1.551850	0.6158
9.862181	0.800000	1.652969	0.6413
10.34752	0.700000	1.721674	0.6692
10.58818	0.600000	1.762157	0.6765
10.49841	0.500000	1.794275	0.6592
10.25000	0.400000	1.823555	0.6370

K	E_i	F_f	x
9.932907	0.300000	1.848083	0.6180
9.568509	0.200000	1.868300	0.6028
9.165277	0.100000	1.885035	0.5911
8.732000	0	1.898509	0.5825
8.273998	− 0.100000	1.909331	0.5764
7.798774	− 0.200000	1.917602	0.5727
7.313147	− 0.300000	1.923475	0.5710
6.819865	− 0.400000	1.927939	0.5709
6.324000	− 0.500000	1.931505	0.5720
5.838000	− 0.600000	1.932889	0.5746
5.360000	− 0.700000	1.934013	0.5779
4.896639	− 0.800000	1.934778	0.5820
4.449041	− 0.900000	1.936347	0.5863
4.021914	− 1.000000	1.938919	0.5906
3.616000	− 1.100000	1.943636	0.5946
3.240000	− 1.200000	1.949489	0.5984
2.890556	− 1.300000	1.958641	0.6012
1.865209	− 1.700000	2.029859	0.6029
1.722888	− 2.000000	2.126496	0.5957
2.731019	− 2.300000	2.256825	0.5932
6.000000	− 2.684600	2.401692	0.6202
6.536376	− 2.800000	2.423280	0.6253
6.708485	− 2.900000	2.436111	0.6261
6.686781	− 3.000000	2.445025	0.6252
6.000000	− 3.280418	2.469526	0.6184
5.000000	− 3.534626	2.489712	0.6147

Class (f)

Initial Conditions: $F_i = 0$; $E_i < 0$; $\dot{E}_i = 0$; $\dot{F}_i > 0$.

Final Conditions: $E_f = 0$; $F_f > 0$; $\dot{F}_f = 0$; $\dot{E}_f > 0$.

Note: This class is also represented by the conditions:

$$\begin{cases} E_i' = 0;\ F_i' > 0;\ \dot{F}_i' = 0;\ \dot{E}_i' > 0. \\ F_f' = 0;\ E_f' > 0;\ \dot{E}_f' = 0;\ \dot{F}_f' < 0. \end{cases}$$

To obtain this representation take $F_i' = F_f$; $E_f' = -E_i$.

K	E_i	F_f	x
12.50000	− 0.922300	+ 0.870369	0.4462
11.25000	− 0.981329	0.920610	0.4646
11.00000	− 0.994345	0.931672	0.4684
10.00000	− 1.051386	0.979800	0.4843
9.000000	− 1.118210	1.035030	0.5008
8.000000	− 1.196873	1.100000	0.5176
6.750000	− 1.319917	1.198307	0.5381
4.844872	− 1.600000	1.417370	0.5636
4.406971	− 1.700000	1.494434	0.5673
4.087715	− 1.800000	1.570804	0.5694
3.822524	− 2.000000	1.724154	0.5711
5.000000	− 2.347730	1.986894	0.5844
7.000000	− 2.596185	2.124250	0.6123
7.500000	− 2.694021	2.153503	0.6192
7.694946	− 2.769554	2.170000	0.6209
4.145615	− 3.860885	2.250000	0.6066
2.957395	− 4.141600	2.252358	0.6135
2.210961	− 4.341600	2.253806	0.6180
1.295156	− 4.641600	2.271379	0.6193

Class (g)

Initial Conditions: $F_i = 0$; $\dot{E}_i = 0$; $\dot{F}_i > 0$.

Final Conditions: $E_f = 0$; $F_f = 0$; $\dot{E}_f < 0$.

Note: This class is also represented by the conditions

$$\begin{cases} E_i' = 0;\ \dot{F}_i' = 0;\ \dot{E}_i' > 0. \\ F_f' = 0;\ \dot{E}_f' = 0;\ \dot{F}_f' > 0. \end{cases}$$

To obtain this representation take $F_i' = -F_f$; $E_f = E_i$.

* These values are due to P. Pedersen (unpublished).

K	E_i	F_f	x
15.77000	1.067147	+ 0.933589	0.7117
15.455	1.100	0.944	*
15.08612	1.150000	+ 0.944618	0.8057
15.02927	1.161819	+ 0.940000	0.8180
14.97895	1.177672	+ 0.930000	0.8326
14.96060	1.189305	+ 0.920000	0.8421
14.956	1.200	0.910	*
15.02675	1.250000	+ 0.843970	0.8820
15.153	1.300	0.766	*
15.28072	1.350000	+ 0.680867	0.9686
15.38760	1.400000	+ 0.582305	1.0455
15.40455	1.410000	+ 0.559673	1.0664
15.41905	1.420000	+ 0.535307	1.0902
15.43004	1.430000	+ 0.508454	1.1179
15.43234	1.432914	+ 0.500000	1.1269
15.43562	1.440000	+ 0.477743	1.1511
15.43073	1.450000	+ 0.440012	1.1936
15.42537	1.453000	+ 0.426212	.2094
15.35624	1.463190	+ 0.350000	1.2955
15.26969	1.464367	+ 0.300000	1.3479
15.00000	1.456591	+ 0.203998	1.4332
14.54460	1.436592	+ 0.100000	1.5038
14.000	1.4104	0.009	*
13.000	1.3578	− 0.125	*
12.50000	1.328125	− 0.178220	1.6355

K	E_i	F_f	x
12.000	1.2949	−0.230	*
11.000	1.2109	−0.317	*
10.196	1.100	−0.363	*
9.915	1.000	−0.344	*
10.0384	0.900	−0.258	*
10.55390	0.810757	−0.062000	2.0179
10.62750	0.804448	−0.005000	2.0298
10.62932	0.804447	−0.001000	2.0294
10.62969	0.804456	−0.000002	2.0293
10.6056	0.810	0.046	*
10.2228	0.900	0.202	*
10.238	1.000	0.330	*
10.738	1.100	0.483	*
11.00000	1.129777	+0.537869	1.4421
11.9487	1.200	0.707	*
12.50000	1.223290	+0.793292	1.2261
13.00000	1.233426	+0.866735	1.1513
13.25000	1.233491	+0.900887	1.1106
13.50000	1.229108	+0.932627	1.0668
13.75000	1.218899	+0.961383	1.0195
14.00000	1.200304	+0.987776	0.9681
14.06565	1.193213	+0.995000	0.9536
14.12277	1.185710	+1.002000	0.9404
14.17037	1.177904	+1.009000	0.9286
14.20704	1.170115	+1.016000	0.9186
14.23303	1.162559	+1.023000	0.9104
14.25198	1.154299	+1.031000	0.9030
14.26518	1.140740	+1.045000	0.8937
14.24737	1.114031	+1.075000	0.8828
14.24022	1.109755	+1.080000	0.8819
14.21033	1.092749	+1.100000	0.8789
14.17500	1.075786	+1.120000	0.8776
14.16475	1.071504	+1.125000	0.8776
14.11490	1.049827	+1.150000	0.8784
14.09440	1.040952	+1.160000	0.8793
14.06313	1.027324	+1.175000	0.8812
14.01000	1.003453	+1.200000	0.8859
13.96863	0.983186	+1.220000	0.8909
13.90000	0.946090	+1.252383	0.9027
13.80000	0.877673	+1.296307	0.9291
13.708	0.800	1.319	*
13.60000	0.716260	+1.317022	0.9824
13.392	0.600	1.285	*
12.8324	0.400	1.186	*
12.50000	0.310989	+1.131696	1.0428
12.0187	0.200	1.053	*
11.00000	0.003349	+0.887609	1.1314
10.3818	−0.100	0.779	*
9.70866	−0.200	0.644	*

K	E_i	F_f	x
9.4	−0.2392	0.570	*
9.02773	−0.200	0.273	*
9.32	−0.100	0.190	*
9.65813	0.000	0.125	*
9.99803	0.100	0.057	*
10.15088	0.146658	0	1.8695
10.16324	0.152940	−0.050000	1.8868
10.08924	0.134355	−0.100000	1.8859
9.959187	0.100271	−0.150000	1.8752
9.958141	0.100000	−0.150372	1.8751
9.792165	0.056413	−0.200000	1.8598
9.575164	0	−0.255922	1.8407
9.18392	−0.100	−0.345	*
8.928545	−0.164434	−0.400000	1.7926
8.787	−0.200	−0.429	*
8.390	−0.300	−0.511	*
7.98	−0.400	−0.589	*
7.622153	−0.50000	−0.666524	1.7232
7.463546	−0.543551	−0.700000	1.7165
6.935505	−0.700000	−0.817336	1.6954
6.391237	−0.900000	−0.960491	1.6752
6.192090	−1.000000	−1.028066	1.6675
5.979706	−1.200000	−1.152400	1.6557
6.072945	−1.400000	−1.262175	1.6452
6.575256	−1.600000	−1.375607	1.6300
7.142005	−1.700000	−1.453193	1.6194
7.600000	−1.744131	−1.500910	1.6148
7.800000	−1.757026	−1.518362	1.6138
8.000000	−1.767032	−1.533985	1.6132
8.400000	−1.780311	−1.560347	1.6134
9.000000	−1.788456	−1.589560	1.6166
9.500000	−1.788293	−1.606238	1.6218
9.800000	−1.786173	−1.613439	1.6261
10.00000	−1.784111	−1.617194	1.6294
10.20000	−1.781566	−1.620155	1.6332
10.40000	−1.778874	−1.622381	1.6377
10.80000	−1.772310	−1.624650	1.6480
11.00000	−1.768701	−1.624735	1.6541
11.50000	−1.758884	−1.621895	1.6731
11.80000	−1.752580	−1.618016	1.6876
12.00000	−1.748238	−1.614457	1.6989
12.25000	−1.742629	−1.608811	1.7154
12.50000	−1.737042	−1.601685	1.7356
12.75000	−1.731457	−1.592836	1.7608
13.25000	−1.719765	−1.568305	1.8367
13.50000	−1.714453	−1.551347	1.9093
13.50063	−1.714650	−1.551300	1.9095
13.54684	−1.713568	−1.547800	1.9298
13.59292	−1.713051	−1.544300	1.9546

K	E_i	F_f	x
13.60000	−1.712944	−1.543770	1.9590
13.64183	−1.712487	−1.540800	1.9895
13.65289	−1.712239	−1.540100	1.9994
13.66530	−1.712168	−1.539400	2.0120
13.68146	−1.712225	−1.538700	2.0313
13.70000	−1.712301	−1.538452	2.0605
13.72000	−1.712617	−1.540099	2.1126
13.72342	−1.712354	−1.541000	2.1279
13.72577	−1.713018	−1.542000	2.1419
13.72730	−1.713303	−1.543000	2.1544
13.72876	−1.713988	−1.545000	2.1765
13.72888	−1.714065	−1.547000	2.1961
13.72737	−1.714736	−1.550000	2.2222
13.72443	−1.715077	−1.553000	2.2458
13.72037	−1.715650	−1.556000	2.2674
13.70953	−1.713700	−1.562000	2.3071
13.70735	−1.716483	−1.563000	2.3124
13.70512	−1.716791	−1.564000	2.3182
13.70041	−1.717565	−1.566000	2.3296
13.69270	−1.720058	−1.569000	2.3455
13.68433	−1.717709	−1.572000	2.3619
13.67517	−1.721161	−1.575000	2.3820
13.66536	−1.719597	−1.578000	2.3885
13.65480	−1.722859	−1.581000	2.4109
13.64360	−1.719845	−1.584000	2.4113
13.63163	−1.723527	−1.587000	2.4339
13.56120	−1.725874	−1.602000	2.4855
13.52785	−1.727897	−1.608000	2.5072
13.49140	−1.731437	−1.614000	2.5330
13.45177	−1.731662	−1.620000	2.5465
13.36236	−1.736543	−1.632000	2.5857
13.13871	−1.746565	−1.656000	2.6692
13.00145	−1.757562	−1.668000	2.7080
12.84481	−1.766647	−1.680000	2.7488
12.66654	−1.772116	−1.692000	2.7926
12.50000	−1.786034	−1.701971	2.8295
12.23370	−1.802228	−1.716000	2.8870
11.97202	−1.809285	−1.728000	2.9438
11.67361	−1.837315	−1.740000	3.0051
11.33152	−1.862229	−1.752000	3.0801
10.93205	−1.893103	−1.764000	3.1795
10.69688	−1.913555	−1.770000	3.2524
10.65300	−1.916950	−1.771000	3.2688
10.49309	−1.932305	−1.774000	3.3476
10.84943	−1.893065	−1.757006	3.7724
10.89800	−1.888436	−1.755483	3.8141
10.93000	−1.885906	−1.754506	3.8454
10.96000	−1.883661	−1.753614	3.8790
10.99000	−1.881730	−1.752751	3.9205

Class (k)

Initial Conditions: $E_i = 0$; $\dot{F}_i = 0$; $\dot{E}_i > 0$.

Final Conditions: $E_f = \pi/2$; $\dot{F}_f = 0$.

Note: The degenerate case of a cusp, with $\dot{E}_f = 0$ and $\ddot{F}_f < 0$, is included.

K	F_i	F_f	x
11.01000	−0.178326	−1.350059	1.7815
11.02000	−0.177874	−1.366491	1.7383
11.03000	−0.176305	−1.382065	1.7015
11.03523	−0.175000	−1.390245	1.6834
11.04000	−0.173452	−1.397921	1.6669
11.04766	−0.170000	−1.411146	1.6395
11.05464	−0.165000	−1.425549	1.6108
11.05800	−0.161067	−1.434723	1.5929
11.05880	−0.159755	−1.437497	1.5875
11.05960	−0.158120	−1.440787	1.5811
11.06004	−0.157000	−1.442979	1.5769
11.06058	−0.155000	−1.446626	1.5698
11.06081	−0.150000	−1.455016	1.5535
11.05708	−0.140000	−1.469248	1.5254
11.04821	−0.130000	−1.481162	1.5006
11.03368	−0.120000	−1.491592	1.4770
11.00000	−0.106920	−1.504330	1.4421
10.90000	−0.104286	−1.514648	1.3748
10.80000	−0.123978	−1.508639	1.3196
10.70000	−0.153306	−1.490708	1.2647
10.57793	−0.250000	−1.394416	1.1308
10.64305	−0.300000	−1.331032	1.0750
10.79181	−0.350000	−1.261394	1.0242
10.90000	−0.375688	−1.223068	0.9995
11.00000	−0.395883	−1.191575	0.9803
11.40000	−0.457584	−1.083667	0.9215
12.00000	−0.514266	−0.942706	0.8580
12.50000	−0.531857	−0.830932	0.8185
14.08304	−0.400000	−0.519540	0.7970
14.47934	−0.300000	−0.468345	0.8485
14.53573	−0.250000	−0.468594	0.8803
14.50197	−0.200000	−0.483197	0.9120
14.38918	−0.150000	−0.509465	0.9419
14.11919	−0.080000	−0.561559	0.9796
14.02093	−0.060000	−0.579180	0.9894
13.67570	0	−0.638425	1.0171
12.94616	0.100000	−0.757791	1.0588
12.50000	0.151334	−0.830558	1.0795
12.02800	0.200000	−0.909943	1.1001
11.50000	0.248068	−1.005200	1.1233
11.00000	0.285817	−1.108344	1.1481
10.70000	0.302165	−1.183163	1.1666

K	F_i	F_f	x
10.40000	0.304584	−1.287343	1.1955
10.30000	0.289585	−1.354040	1.2177
10.28540	0.270000	−1.397403	1.2351
10.30476	0.250000	−1.427747	1.2495
10.35053	0.225000	−1.455818	1.2657
10.40867	0.200000	−1.477184	1.2809
10.54233	0.150000	−1.506776	1.3110
10.80000	0.060258	−1.529486	1.3744
10.90000	0.028902	−1.527984	1.4099
10.95000	0.017594	−1.524041	1.4341
11.00000	0.013423	−1.516528	1.4672
11.04445	0.020000	−1.502849	1.5127
11.06615	0.030000	−1.488197	1.5523
11.07498	0.040000	−1.472621	1.5886
11.07517	0.050000	−1.454774	1.6289
11.06716	0.060000	−1.433368	1.6705

Class (l)

Initial Conditions: Same as class (k)

Final Conditions: " " " "

Note: Same as class (k)

K	F_i	F_f	x
15.00000	2.420000	2.432235	0.0623
14.43600	2.292000	2.306153	0.0854
14.00000	2.198500	2.209775	0.1090
13.62360	2.108000	2.111160	0.1409
13.40000	2.055000	2.046758	0.1665
13.18580	2.010000	1.982502	0.1955
13.00000	1.979181	1.926679	0.2227
12.80000	1.956629	1.869102	0.2515
12.60800	1.944000	1.817073	0.2771
12.50000	1.939506	1.788699	0.2911
12.20000	1.935192	1.713904	0.3265
12.00000	1.936491	1.665592	0.3486
11.76700	1.940707	1.609391	0.3740
11.60000	1.945092	1.568347	0.3926
11.44620	1.950000	1.529342	0.4110
11.19308	1.960000	1.460643	0.4433
11.00000	1.969582	1.401354	0.4735
10.99311	1.970000	1.399026	0.4748
10.85900	1.978400	1.350371	0.5020
10.70000	1.993045	1.270456	0.5547
10.66017	2.000000	1.233761	0.5837
10.65713	2.010000	1.181805	0.6363
10.69481	2.015000	1.155900	0.6747
10.75000	2.018090	1.139731	0.7119
10.83617	2.020000	1.128450	0.7645
11.00000	2.016864	1.149267	0.8957
11.02655	2.015000	1.165802	0.9409
11.04097	2.013000	1.189138	0.9967
11.04222	2.012000	1.205102	1.0330

Class (m)

Initial Conditions: $E_i = 0$; $\dot{F}_i = 0$; $\dot{E}_i > 0$.

Final Conditions: $E_f = \pi/2$; $\dot{F}_f = 0$; $\dot{E}_f > 0$.

K	F_i	F_f	x
−385.9430	0.010000	0.051217	0.1504
−68.99444	0.050000	0.123359	0.2462
−30.33587	0.100000	0.188305	0.3301
−11.87477	0.200000	0.301149	0.3301
−6.340919	0.300000	0.406675	0.3433
−4.009862	0.400000	0.509140	0.3448
−3.376365	0.450000	0.559638	0.3429
−2.955235	0.500000	0.609748	0.3395
−2.535816	0.600000	0.708947	0.3293
−2.475000	0.700000	0.806923	0.3155
−2.471274	0.650000	0.758080	0.3228
−2.633037	0.800000	0.903772	0.2990
−2.931662	0.900000	0.999577	0.2805
−3.324117	1.000000	1.094444	0.2605
−3.781301	1.100000	1.188251	0.2398
−4.284452	1.200000	1.281980	0.2187
−5.385719	1.400000	1.467879	0.1775
−6.576374	1.600000	1.653771	0.1397
−7.838336	1.800000	1.840946	0.1069
−9.169998	2.000000	2.030153	0.0798

Class (n)

Initial Conditions: $F_i = 0$; $\dot{E}_i = 0$; $\dot{F}_i > 0$.

Final Conditions: $F_f = 0$; $\dot{E}_f = 0$; $\dot{F}_f < 0$.

Note: To obtain the remainder of the class take the mirror images, i.e. the values $E_i' = -E_f$; $E_f' = -E_i$.

K	E_i	E_f	x
4.870000	−1.594500	1.595414	1.1268
4.873414	−1.540000	1.649716	1.1266
4.886110	−1.500000	1.686880	1.1262
4.944520	−1.400000	1.774663	1.1244
4.988954	−1.350000	1.815711	1.1233
5.043931	−1.300000	1.854939	1.1220
5.109578	−1.250000	1.892482	1.1207

K	E_i	E_f	x
5.184459	−1.200000	1.929580	1.1200
5.600000	−1.000000	2.057225	1.1158
6.510801	−0.700000	2.209606	1.1193
6.872378	−0.600000	2.252829	1.1239
7.646494	−0.400000	2.333213	1.2292
8.449144	−0.200000	2.398000	1.1649
10.37	0.44627	2.69532	1.2931
6.038574	1.000000	3.984834	1.1159

Class (α)

Initial Conditions: $E_i = 0$; $\dot{F}_i = 0$; $\dot{E}_i > 0$.

Final Conditions: $E_f = 0$; $\dot{F}_f = 0$; $\dot{E}_f > 0$.

Note: To obtain the remainder of the class take the mirror images, i.e. the values

$$F_i' = F_f; \quad F_f' = F_i$$

K	F_i	F_f	x
10.41790	0.363000	1.774431	4.0088
10.45000	0.360288	1.773643	3.9572
10.50000	0.356883	1.773962	3.9182
10.74655	0.340000	1.769071	3.8058
11.00000	0.320953	1.762218	3.7233
11.01214	0.320000	1.762063	3.7196
11.25806	0.300000	1.754475	3.6509
11.49058	0.280000	1.746746	3.5907
11.71296	0.260000	1.738651	3.5356
11.92706	0.240000	1.730072	3.4835
12.13401	0.220000	1.720825	3.4334
12.33456	0.200000	1.711073	3.3841
12.50000	0.183040	1.702062	3.3423
12.71843	0.160000	1.688806	3.2847
12.90255	0.140000	1.675898	3.2326
13.08205	0.120000	1.661562	3.1767
13.25767	0.100000	1.644305	3.1147
13.34444	0.090000	1.634060	3.0799
13.43100	0.080000	1.623298	3.0398
13.51803	0.070000	1.609943	2.9925
13.61000	0.059724	1.592900	2.9263
13.66490	0.054000	1.579334	2.8702
13.71001	0.050000	1.564487	2.7979
13.74582	0.050000	1.540588	2.6370
13.74000	0.053543	1.533913	2.5627
13.72000	0.059071	1.531315	2.5028
13.70000	0.063587	1.531491	2.4709
13.65000	0.073612	1.534261	2.4228
13.60000	0.082780	1.538146	2.3920

K	F_i	F_f	x
13.50000	0.099944	1.545679	2.3510
13.00000	0.175591	1.574852	2.2570
12.50000	0.243316	1.592654	2.2138
12.00000	0.307075	1.602831	2.1872
11.00000	0.429593	1.605159	2.1556
10.00000	0.553587	1.582818	2.1369
9.500000	0.619567	1.559595	2.1297
9.000000	0.691425	1.524161	2.1231
8.750000	0.730731	1.500746	2.1199
8.500000	0.773761	1.471369	2.1168
8.000000	0.876841	1.390572	2.1109
7.500000	1.072182	1.205712	2.1062
7.450000	1.140000	1.140444	2.1070

Class (β)

Initial Conditions: $F_i = 0$; $\dot{E}_i = 0$; $\dot{F}_i > 0$.

Final Conditions: $E_f = 0$; $\dot{F}_f = 0$; $\dot{E}_f > 0$.

Note: This class is also represented by the conditions

$$\begin{cases} E_i' = 0; \; \dot{F}_i' = 0; \; \dot{E}_i' > 0. \\ F_f' = 0; \; \dot{E}_f' = 0; \; \dot{F}_f' < 0. \end{cases}$$

To obtain this representation take

$$F_i' = F_f; \quad E_f' = -E_i$$

K	E_i	F_f	x
11.00000	0.296314	1.753452	4.2909
10.99843	0.296000	1.753703	4.2207
10.99454	0.295000	1.753816	4.0978
10.99134	0.294000	1.753643	4.0165
10.98855	0.293000	1.753767	3.9496
10.98615	0.292000	1.754083	3.8894
10.98418	0.291000	1.754138	3.8314
10.98276	0.290000	1.754080	3.7715
10.98238	0.289000	1.754105	3.7009
10.98500	0.288304	1.754146	3.6085
11.00000	0.289930	1.753872	3.4587
11.00041	0.290000	1.753428	3.4563
11.01519	0.293000	1.753391	3.3766
11.01952	0.294000	1.753135	3.3573
11.02765	0.296000	1.752992	3.3238
11.05588	0.304000	1.752073	3.2249
11.07956	0.312000	1.751032	3.1511
11.09961	0.320000	1.752280	3.0873
11.11615	0.328000	1.749572	3.0311
11.12700	0.334737	1.749202	2.9844

K	E_i	F_f	x	K	E_i	F_f	x
11.13306	0.340000	1.748841	2.9476	13.62502	0.840000	1.602303	1.4603
11.13630	0.348000	1.748476	2.8871	13.62763	0.860000	1.601339	1.4549
11.13000	0.353563	1.748602	2.8342	13.62680	0.880000	1.601025	1.4501
11.11000	0.356919	1.749002	2.7756	13.62238	0.900000	1.601416	1.4461
11.08351	0.356500	1.749249	2.7313	13.61434	0.920000	1.601888	1.4439
11.06585	0.355000	1.749670	2.7089	13.60206	0.940000	1.604200	1.4419
11.04161	0.352000	1.750291	2.6832	13.56433	0.980000	1.609806	1.4445
11.00000	0.345162	1.751438	2.6472	13.53800	1.000000	1.612681	1.4497
10.96849	0.339000	1.752424	2.6244	13.50500	1.020000	1.617975	1.4559
10.90983	0.326000	1.754746	2.5881	13.46470	1.040000	1.623569	1.4651
10.85652	0.313000	1.755470	2.5609	13.41529	1.060000	1.629622	1.4777
10.80620	0.300000	1.756991	2.5379	13.35379	1.080000	1.636688	1.4940
10.73217	0.280000	1.759193	2.5082	13.27528	1.100000	1.645197	1.5148
10.48665	0.210000	1.766784	2.4314	13.17072	1.120000	1.655222	1.5423
10.24729	0.140000	1.774544	2.3749	13.01799	1.140000	1.668439	1.5807
10.00932	0.070000	1.782501	2.3283	12.90000	1.150327	1.677341	1.6091
9.774416	0.000002	1.791323	2.2863	12.80000	1.156523	1.684432	1.6322
9.675929	−0.030000	1.794698	2.2691	12.60000	1.163673	1.697989	1.6759
9.548671	−0.070000	1.800360	2.2451	12.50000	1.165269	1.702867	1.6977
9.428878	−0.110000	1.805064	2.2198	12.20000	1.163870	1.719483	1.7587
9.323869	−0.150000	1.812061	2.1880	11.80000	1.151349	1.735593	1.8390
9.281781	−0.170000	1.815022	2.1684	11.60000	1.141187	1.743291	1.8790
9.254982	−0.190000	1.818156	2.1421	11.40000	1.128500	1.748700	1.9208
9.270000	−0.207453	1.821542	2.1016	11.30000	1.121059	1.753599	1.9412
9.290000	−0.210817	1.820845	2.0869	11.20000	1.112896	1.756690	1.9629
9.340000	−0.212568	1.820671	2.0611	11.00000	1.093879	1.762680	2.0088
9.440000	−0.208197	1.819397	2.0242	10.80000	1.070102	1.768068	2.0598
9.640000	−0.189526	1.815530	1.9692	10.70018	1.055600	1.770559	2.0886
9.840000	−0.165474	1.810431	1.9250	10.65000	1.047400	1.770046	2.1052
10.02943	−0.140000	1.805746	1.8882	10.60000	1.038270	1.772908	2.1212
10.56000	−0.060000	1.798043	1.8025	10.50000	1.016265	1.774855	2.1607
10.92429	0	1.780024	1.7515	10.43580	0.997071	1.775997	2.1936
11.00000	0.013034	1.777580	1.7418	10.43562	0.997000	1.776002	2.1937
11.04000	0.020000	1.776537	1.7367	10.38694	0.975000	1.776447	2.2306
11.47957	0.100000	1.761059	1.6852	10.36263	0.950000	1.776051	2.2722
11.88240	0.180000	1.747610	1.6420	10.36970	0.926000	1.774716	2.3129
12.24890	0.260000	1.728032	1.6088	10.41139	0.900000	1.772659	2.3594
12.50000	0.320434	1.714170	1.5873	10.50676	0.870000	1.768379	2.4212
12.57639	0.340000	1.709719	1.5809	10.60514	0.850000	1.764910	2.4730
12.86304	0.420000	1.690604	1.5581	10.75450	0.830000	1.759856	2.5516
13.10698	0.500000	1.668635	1.5407	10.81456	0.825000	1.757207	2.5872
13.21189	0.540000	1.661300	1.5307	10.83000	0.824008	1.755278	2.5976
13.38780	0.620000	1.642144	1.5144	10.84942	0.823000	1.756707	2.6092
13.45820	0.660000	1.633121	1.5059	10.87000	0.822133	1.752688	2.6250
13.51687	0.700000	1.623098	1.4980	10.89800	0.821500	1.755133	2.6438
13.56333	0.740000	1.615229	1.4883	10.90000	0.821500	1.757126	2.6443
13.58190	0.760000	1.612215	1.4825	10.93000	0.821504	1.754201	2.6697
13.61000	0.800000	1.605204	1.4725	10.96000	0.822397	1.753314	2.6971
13.61910	0.820000	1.603748	1.4661	10.99000	0.824393	1.752569	2.7291

K	E_i	F_f	x
11.00000	0.825361	1.752191	2.7413
11.02000	0.827909	1.741727	2.7686
11.03243	0.830000	1.751292	2.7885
11.04216	0.832000	1.751064	2.8061
11.05033	0.834000	1.750917	2.8230
11.05725	0.836000	1.750610	2.8396
11.06314	0.838000	1.750551	2.8557
11.06575	0.839000	1.750455	2.8637
11.07236	0.842000	1.750232	2.8878
11.07736	0.845000	1.750350	2.9118
11.08257	0.850000	1.749694	2.9537
11.08410	0.855000	1.750051	2.9982
11.08193	0.860000	1.750648	3.0478
11.06314	0.870000	1.751375	3.1831
11.05300	0.872566	1.752803	3.2367
11.02500	0.876159	1.752910	3.3877
11.00000	0.875772	1.753818	3.5791

Class (γ)

Initial Conditions: same as class (n)
Final Conditions: " " " "
Note: " " " "

K	E_i	E_f	x
10.90000	0.817971	0.056915	3.1730
10.85000	0.819273	0.065338	3.1405
10.83200	0.820131	0.068339	3.1298
10.80000	0.822103	0.073621	3.1119
10.78772	0.823000	0.075635	3.1053
10.70404	0.831000	0.089144	3.0638
10.58929	0.847000	0.107084	3.0121
10.43938	0.879000	0.129144	2.9425
10.35416	0.911000	0.140549	2.8873
10.32287	0.943000	0.143254	2.8361
10.34358	0.975000	0.137866	2.7847
10.41720	1.007000	0.124167	2.7306
10.54691	1.039000	0.101503	2.6721
10.73882	1.071000	0.068107	2.6073
10.86153	1.087000	0.046619	2.5718
10.93053	1.095000	0.034432	2.5531
11.00000	1.102467	0.022067	2.5349
11.00516	1.103000	0.021146	2.5336
11.04475	1.107000	0.014054	2.5236
11.08597	1.111000	0.006639	2.5133
11.10721	1.113000	0.002798	2.5081
11.12890	1.115000	−0.001127	2.5028
11.15102	1.117000	−0.005164	2.4975
11.17365	1.119000	−0.009274	2.4920
11.19675	1.121000	−0.013496	2.4865
11.24449	1.125000	−0.022262	2.4753
11.29442	1.129000	−0.031492	2.4637
11.32026	1.131000	−0.036297	2.4578
11.34674	1.133000	−0.041236	2.4518
11.36021	1.134000	−0.043759	2.4487
11.37386	1.135000	−0.046318	2.4456
11.38768	1.136000	−0.048917	2.4425
11.41586	1.138000	−0.054227	2.4362
11.45000	1.140354	−0.060694	2.4286
11.50000	1.143670	−0.070231	2.4176
11.75000	0.119148	−1.158031	2.3638
12.00000	0.170468	−1.168722	2.3115
12.25000	1.175636	−0.224769	2.2595
12.50000	1.178148	−0.282896	2.2068
12.75000	1.174909	−0.346100	2.1521
13.00000	1.163301	−0.416443	2.0941
13.23527	1.140000	−0.492650	2.0347
13.25000	1.137959	−0.497871	2.0307
13.35851	1.120000	−0.538639	2.0010
13.44913	1.100000	−0.576578	1.9749
13.60106	1.050000	−0.653522	1.9286
13.69351	1.000000	−0.717404	1.8988
13.74940	0.950000	−0.774683	1.8803
13.77771	0.900000	−0.828037	1.8708
13.77800	0.860000	−0.859803	1.8725

Class (δ)

Initial Conditions: same as class (a)
Final Conditions: " " " "
Note: " " " "

K	F_i	F_f	x
15.42751	−0.431093	0.431164	2.4075
15.42472	−0.450000	0.411502	2.4087
15.39431	−0.500000	0.363021	2.4110
15.38426	−0.510000	0.353558	2.4117
15.33127	−0.550000	0.316085	2.4158
15.23746	−0.600000	0.270549	2.4232
15.11434	−0.650000	0.226170	2.4336
14.96243	−0.700000	0.182592	2.4477
14.77992	−0.750000	0.139089	2.4665
14.55881	−0.800000	0.094146	2.4928
14.26612	−0.850000	0.042882	2.5347
13.98826	−0.878648	0	2.5824
13.50000	−0.882014	−0.066433	2.6786
13.00000	−0.840666	−0.126677	2.7782
12.50000	−0.777582	−0.181433	2.8719

K	F_i	F_f	x	K	F_i	F_f	x
12.25000	−0.741723	−0.207122	2.9171	9.060156	2.160000	−1.615732	1.2587
12.00000	−0.703796	−0.231782	2.9620	8.993231	2.155000	−1.582152	1.2545
11.75000	−0.663960	−0.255441	3.0072	8.936470	2.150000	−1.540651	1.2499
11.50000	−0.622102	−0.278051	3.0534	8.920393	2.148000	−1.520819	1.2481
11.25000	−0.577804	−0.299544	3.1018	8.911194	2.146000	−1.498221	1.2464
11.00000	−0.530173	−0.319742	3.1541	8.880000	2.145000	−1.494412	1.2430
10.75000	−0.477313	−0.338323	3.2131	8.900000	2.143993	−1.475326	1.2439
10.50000	−0.414263	−0.354717	3.2860	8.918614	2.143000	−1.456629	1.2444
10.37803	−0.375000	−0.361671	3.3329	8.930000	2.142000	−1.439750	1.2440
10.34761	−0.363293	−0.363293	3.3471	8.950259	2.141000	−1.420101	1.2439
10.31673	−0.350000	−0.364907	3.3634	8.982443	2.140000	−1.396882	1.2442
10.23915	−0.300000	−0.369171	3.4254	9.000000	2.138922	−1.376527	1.2428
10.23082	−0.250000	−0.371252	3.4891	9.200000	2.137409	−1.301869	1.2504
10.29758	−0.200000	−0.370589	3.5571	9.400000	2.136243	−1.240093	1.2579
10.43171	−0.150000	−0.365520	3.6335	9.600000	2.135506	−1.194433	1.2691
10.53000	−0.120716	−0.360430	3.6825	9.800000	2.134132	−1.149678	1.2812
10.63000	−0.091068	−0.354481	3.7334	10.20000	2.129964	−1.077203	1.3098
10.73000	−0.052594	−0.347941	3.7897	10.40000	2.126529	−1.043404	1.3272
10.75000	−0.038431	−0.346567	3.8034	10.56250	2.122914	−1.017429	1.3433
10.75878	−0.024200	−0.345972	3.8116	10.70000	2.119155	−0.996408	1.3587
10.74980	0	−0.346634	3.8138	10.80000	2.115905	−0.980988	1.3706
10.70000	0.034528	−0.350059	3.8002	11.00000	2.107948	−0.952129	1.3987
10.60000	0.079294	−0.356577	3.7715	11.20000	2.097267	−0.924341	1.4335
10.50000	0.116703	−0.362488	3.7457	11.40000	2.082315	−0.898337	1.4808
10.40000	0.151271	−0.367676	3.7228	11.58879	2.060000	−0.876191	1.5527
10.30000	0.184561	−0.371880	3.7021	11.67662	2.040000	−0.868060	1.6239
10.20000	0.217621	−0.374714	3.6833	11.68967	2.020000	−0.870892	1.7115
10.10000	0.251672	−0.375306	3.6660	11.57100	2.000000	−0.894286	1.8526
10.05000	0.269742	−0.374176	3.6579	11.46985	1.995000	−0.911587	1.9266
10.00000	0.289358	−0.371286	3.6500	11.40000	1.993425	−0.923328	1.9736
9.950000	0.312682	−0.364455	3.6425	11.30000	1.992877	−0.939657	2.0396
9.917145	0.338570	−0.349332	3.6376	11.29000	1.992919	−0.940412	2.0462
9.915198	0.343950	−0.343877	3.6381	11.28500	1.992937	−0.942038	2.0497

Class (λ)

Initial Conditions: same as class (k)
Final Conditions: „ „ „ „
Note: „ „ „ „

K	F_i	F_f	x
8.100000	2.249498	−1.822197	1.1933
8.500000	2.236850	−1.810056	1.2068
8.678465	2.230000	−1.802585	1.2146
8.900000	2.220000	−1.789342	1.2268
9.071346	2.210000	−1.774346	1.2389
9.193476	2.200000	−1.756117	1.2502
9.258044	2.190000	−1.733862	1.2592
9.228497	2.175000	−1.687927	1.2652
9.126486	2.165000	−1.643750	1.2622

Class (μ)

Initial Conditions: same as class (a)
Final Conditions: „ „ „ „
Note: „ „ „ „

K	F_i	F_f	x
11.00000	0.072157	−2.012387	3.2029
10.96670	0.080000	−2.012861	2.8931
10.94752	0.090000	−2.011901	2.7397
10.94207	0.097000	−2.010610	2.6460
10.95000	0.104343	−2.008244	2.5314
11.00000	0.106976	−2.004080	2.3792
11.10000	0.095038	−2.000261	2.2332
11.20000	0.072707	−1.998959	2.1256
11.26799	0.050000	−1.999451	2.0562

K	F_i	F_f	x
11.35156	0	−2.002502	1.9589
11.37887	−0.050000	−2.005166	1.9070
11.37226	−0.100000	−2.005713	1.8912
11.33441	−0.150000	−2.003950	1.9137
11.28554	−0.180000	−2.002158	1.9550
11.22624	−0.200000	−2.001050	2.0098
11.17276	−0.210000	−2.000880	2.0633
11.06205	−0.215000	−2.002684	2.1991
11.00000	−0.208417	−2.005272	2.3131
10.98000	−0.203751	−2.006554	2.3679
10.96000	−0.195771	−2.008424	2.4532
10.95155	−0.180000	−2.011315	2.6320
10.96000	−0.172754	−2.012284	2.7424
11.00000	−0.166372	−2.012276	3.1527

Class (v)

Initial Conditions: $E_i = 0$; $\dot{F}_i = 0$; $\dot{E}_i > 0$.

Final Conditions: $E_f = -\pi/2$; $\dot{F}_f = 0$.

K	F_i	F_f	x
11.00000	1.752480	−0.982134	3.3688
10.75806	1.760000	−1.022996	3.1371
10.61518	1.765000	−1.049386	3.0433
10.49074	1.770000	−1.075521	2.9534
10.45183	1.772000	−1.084942	2.9103
10.45000	1.774192	−1.090898	2.8192
10.50000	1.773844	−1.084276	2.7731
10.59500	1.772193	−1.069324	2.7226
10.60000	1.772092	−1.068514	2.7204
10.70000	1.769894	−1.052050	2.6817
10.80000	1.767457	−1.035581	2.6492
10.90000	1.764844	−1.019031	2.6201
11.00000	1.762084	−1.002808	2.5935
11.20000	1.756160	−0.970468	2.5446
11.40000	1.749712	−0.938799	2.4996
11.60000	1.742710	−0.907601	2.4569
11.80000	1.735091	−0.876836	2.4155
12.00000	1.726767	−0.846368	2.3746
12.40000	1.707465	−0.785985	2.2919
12.50000	1.701947	−0.770908	2.2705
12.60000	1.696085	−0.755790	2.2485
12.80000	1.683133	−0.725442	2.2023
13.00000	1.668078	−0.694761	2.1517
13.20000	1.650008	−0.663420	2.0932
13.40000	1.627072	−0.630860	2.0195
13.57045	1.600000	−0.600929	1.9259
13.65645	1.580000	−0.583595	1.8430
13.68727	1.570000	−0.576119	1.7924
13.70920	1.560000	−0.569200	1.7312
13.71604	1.555000	−0.565825	1.6942
13.71863	1.550000	−0.562346	1.6495
13.71761	1.548000	−0.560850	1.6277
13.71500	1.546337	−0.559489	1.6063
13.71000	1.544887	−0.558120	1.5825
13.70000	1.543742	−0.556625	1.5527
13.60000	1.548590	−0.554514	1.4400
13.40000	1.564793	−0.559820	1.3573
13.20000	1.579213	−0.566901	1.3124
13.00000	1.591743	−0.573990	1.2818
12.80000	1.602777	−0.580631	1.2590
12.60000	1.612627	−0.586659	1.2413
12.50000	1.617182	−0.589415	1.2339
12.40000	1.621523	−0.591987	1.2272
12.20000	1.629641	−0.596568	1.2159
12.00000	1.637116	−0.600359	1.2068
11.80000	1.644059	−0.603328	1.1995
11.40000	1.656704	−0.606629	1.1892
11.00000	1.668206	−0.605944	1.1834
10.60000	1.679115	−0.600476	1.1815
10.20000	1.690028	−0.588675	1.1827
9.800000	1.701749	−0.567314	1.1867
9.600000	1.708336	−0.550703	1.1896
9.400000	1.715865	−0.526995	1.1930
9.200000	1.725406	−0.487529	1.1966
9.130168	1.730000	−0.463103	1.1975
9.081897	1.735000	−0.429781	1.1972
9.081479	1.740000	−0.383136	1.1945
9.122139	1.742500	−0.347436	1.1908
9.200000	1.744162	−0.307684	1.1853
9.300000	1.744806	−0.269995	1.1790
9.400000	1.744753	−0.238443	1.1732
9.600000	1.743611	−0.184157	1.1625
10.00000	1.739266	−0.091486	1.1435
10.40000	1.733379	−0.006948	1.1276
10.80000	1.726334	0.076240	1.1150
11.20000	1.718187	0.162890	1.1065
11.60000	1.708926	0.259121	1.1041
12.00000	1.698768	0.378954	1.1138
12.20000	1.693883	0.463593	1.1300
12.30000	1.692130	0.527666	1.1477
12.35000	1.692281	0.585255	1.1673
12.35000	1.696160	0.677680	1.2055
12.30000	1.700787	0.736167	1.2334
12.20000	1.707652	0.802294	1.2678
12.10000	1.713502	0.850924	1.2949
12.00000	1.718794	0.892170	1.3190

K	F_i	F_f	x	K	F_i	F_f	x
11.80000	1.728260	0.963810	1.3631	10.79000	1.761676	1.354741	1.6705
11.60000	1.736633	1.028624	1.4056	10.78500	1.761644	1.364331	1.6814
11.40000	1.744152	1.091488	1.4493	10.78500	1.761034	1.394146	1.7194
11.20000	1.750925	1.156344	1.4969	10.80000	1.760178	1.412335	1.7479
11.00000	1.756943	1.229411	1.5544	10.90000	1.756287	1.446734	1.8477
10.80000	1.761610	1.341783	1.6566	11.00000	1.753122	1.439686	1.9729

Simple asymptotic-periodic (limiting) half-orbits
symmetric with respect to the E- or F-axis.

$K = 11.0$ $\quad\quad E_i = -\pi/2.$

$\dot{F}_i$ has the same sign as $\dot{E}_i$ for a given orbit.

i = that intercept of the line $E = E_f$ at which $\dot{F}_f = 0$ is satisfied.

j = that intercept of the line $F = F_f$ at which $\dot{E}_f = 0$ is satisfied.

Orbit	F_i	$\dot{E}_i$	E_f	i	F_f	j	Simple classes involved
I	1.307735	−0.046274	$-\pi$	1	0.17150		k, μ
II	1.306357	−0.053104	$-\pi$	1	−0.07653		k, μ
III	1.319335	0.012090	−0.87056		0	1	β, γ
IV	1.321045	0.020829	−0.29566		0	1	β
V	1.324681	0.039521	0	1	2.0115		l, μ
VI	1.323609	0.033997	0	2	1.0030		
VII	1.310836	−0.030828	0	2	1.7538		g, α, β, ν
VIII	1.301025	−0.079346	−1.883		0	2	g
IX	1.328004	0.056745	π	1	2.4646		
X	1.311214	−0.028938	0	1	2.3122		
XI	1.322852	0.030100	0	3	0.4065		α
XII	1.310439	−0.032815	−3.10431		0	2	γ
XIII	1.321178	0.021508	$-\pi$	1	−1.9901		λ
XIV	1.322044	0.025964	0	3	−0.81235		

The values of η_i are, respectively:

I, 1.713681; II, 1.710949; III, 1.736810; IV, 1.740240; V, 1.747549; VI, 1.745392; VII, 1.719839; VIII, 1.700403; IX, 1.754245; X, 1.720592; XI, 1.743870; XII, 1.719051; XIII, 1.740506; XIV, 1.742247.

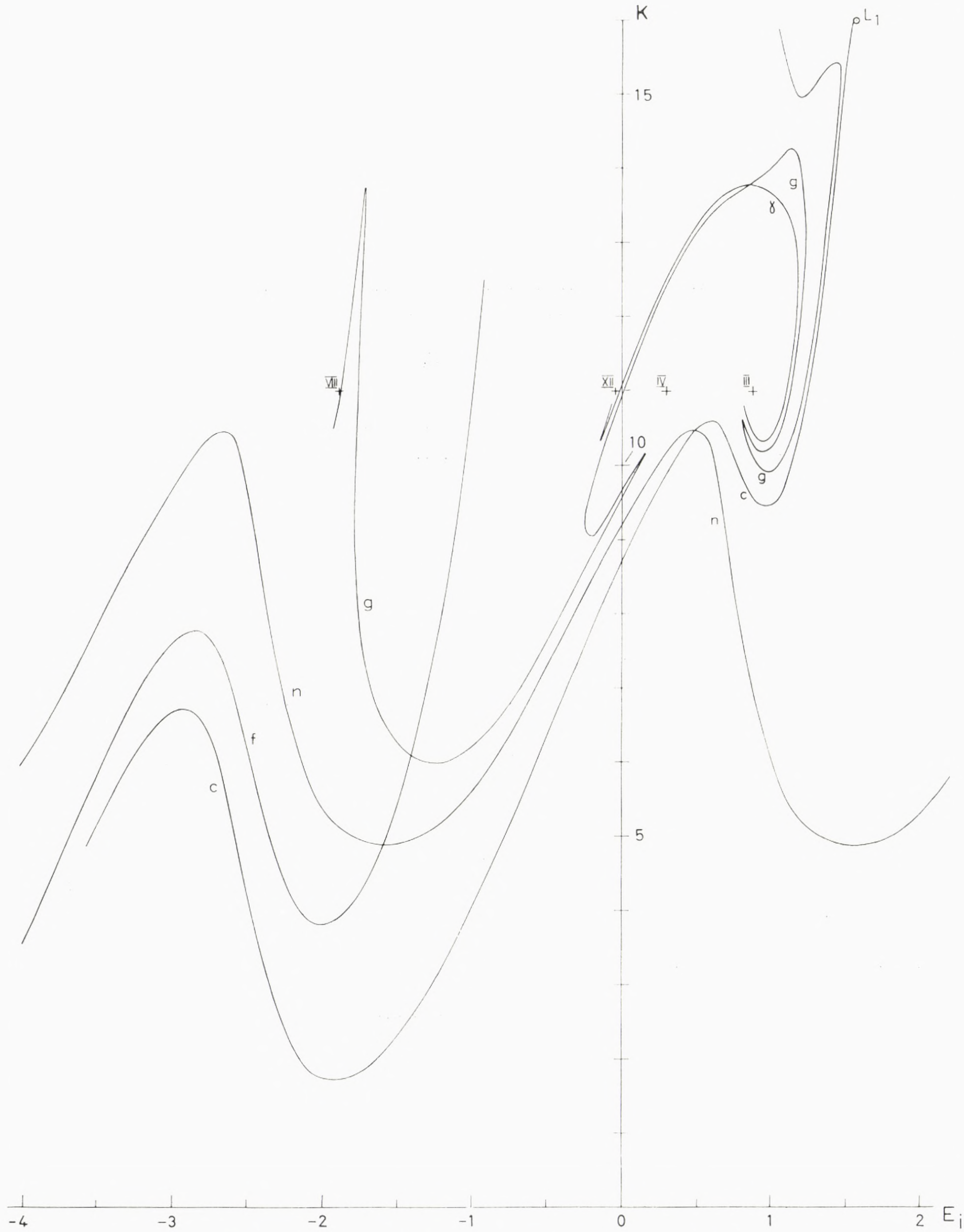

Figure 1 a: General Profile of Eigensurfaces, K vs E_i.
A cross (+) with a Roman numeral, at $K = 11.0$, represents a limiting orbit, shown in detail in Figure 10.

Figure 1 b: Detailed Profile, K vs E_i, showing the (β), (γ) and (g) classes. The crosses (+) at $K = 11.0$ represent limiting orbits.

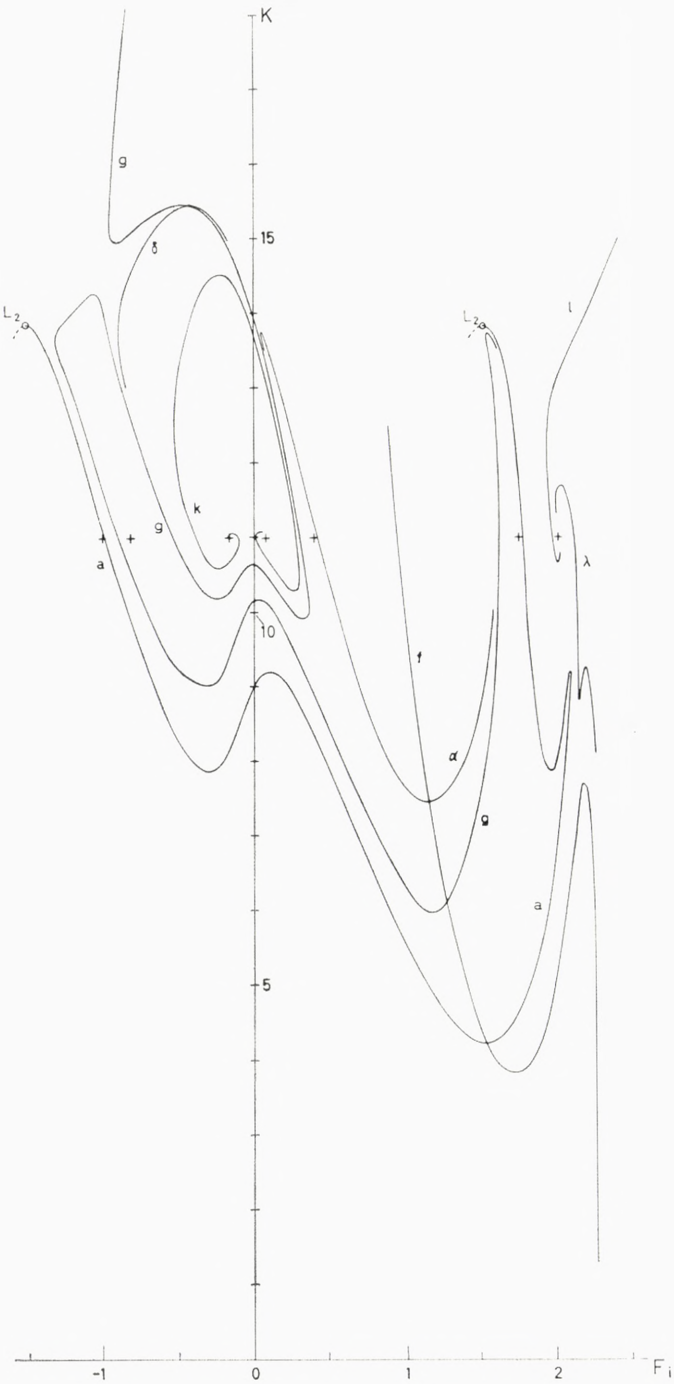

Figure 2 a: General Profile of Eigensurfaces, K vs F_i.
The crosses (+) at $K = 11.0$ represent limiting orbits. (Mr. C. Wagner first noticed the difference between classes λ and l).

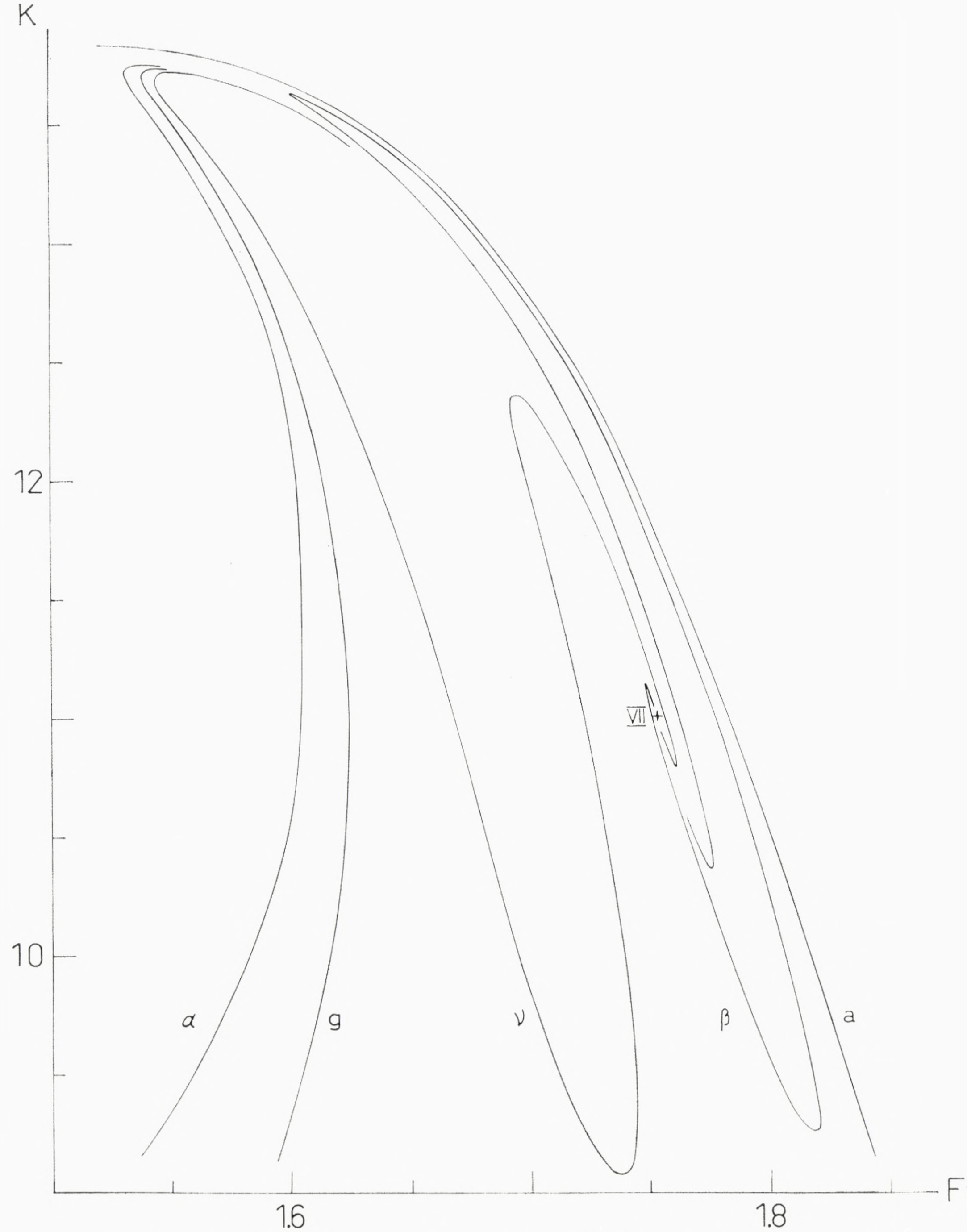

Figure 2 b: Detailed Profile, K vs F_i, showing the (α), (g), (ν), (β) and (a) classes.

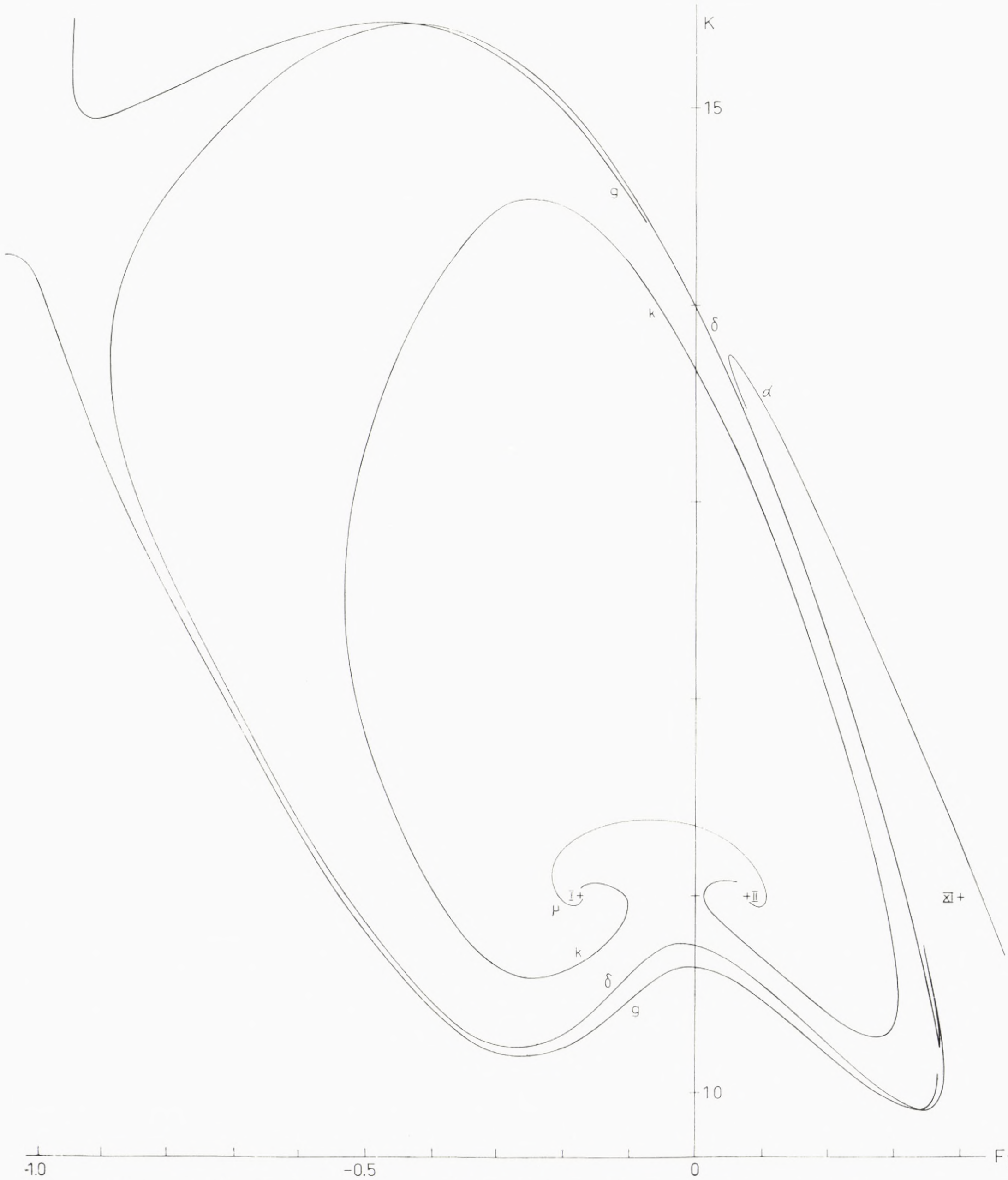

Figure 2 c: Detailed Profile, K vs F_i, showing the (g), (δ), (k), (μ), and (α) classes. The crosses $(+)$ at $K = 11.0$ represent limiting orbits.

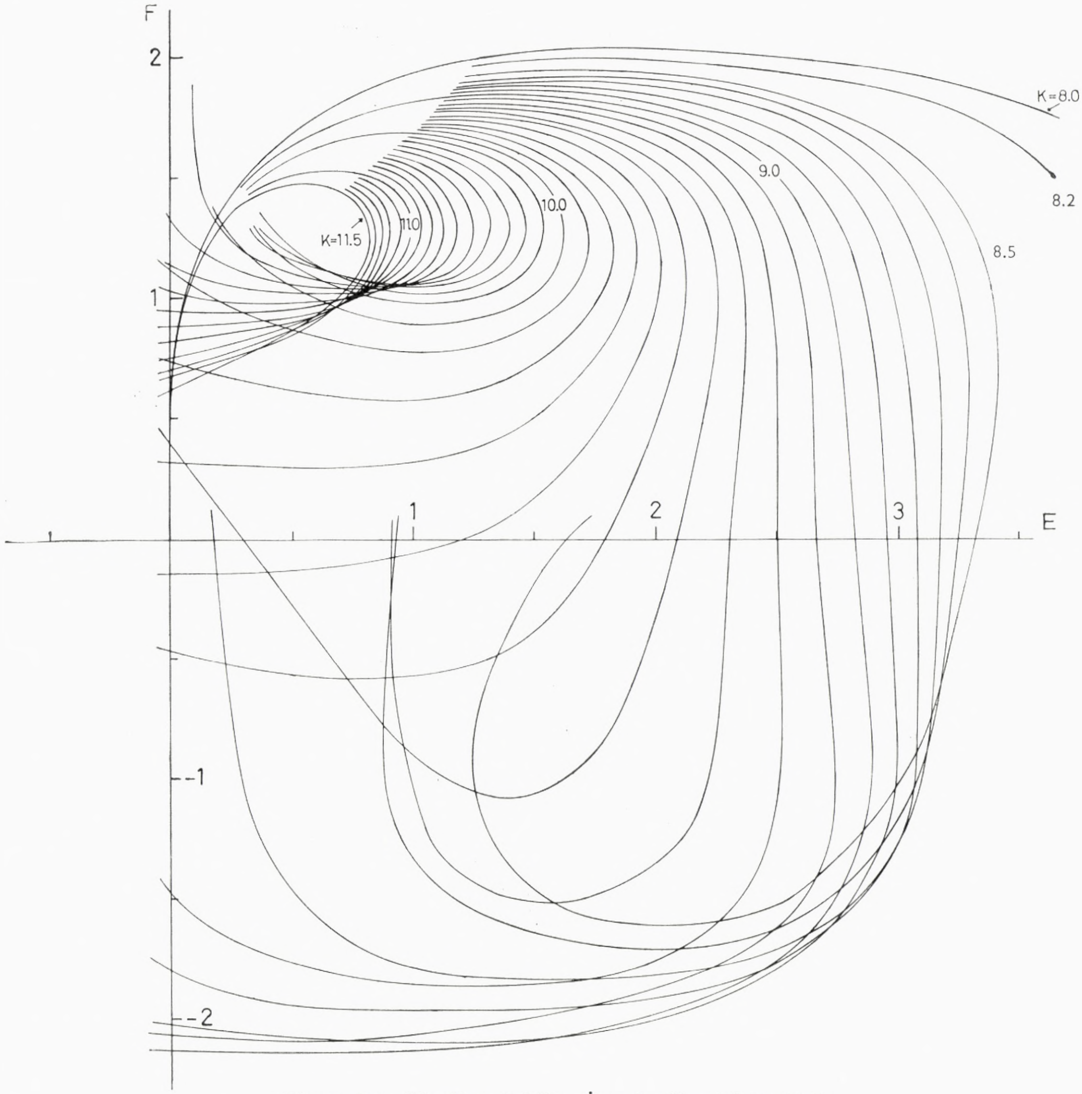

Figure 3 a: Ejection Orbits, $\dot{E}_i = 0$, $K = 8.0$ to 11.5.
Where the values of the Jacobi integral are not explicitly given, the increment from one curve to the next is constant. This remark holds for the subsequent figures, also.

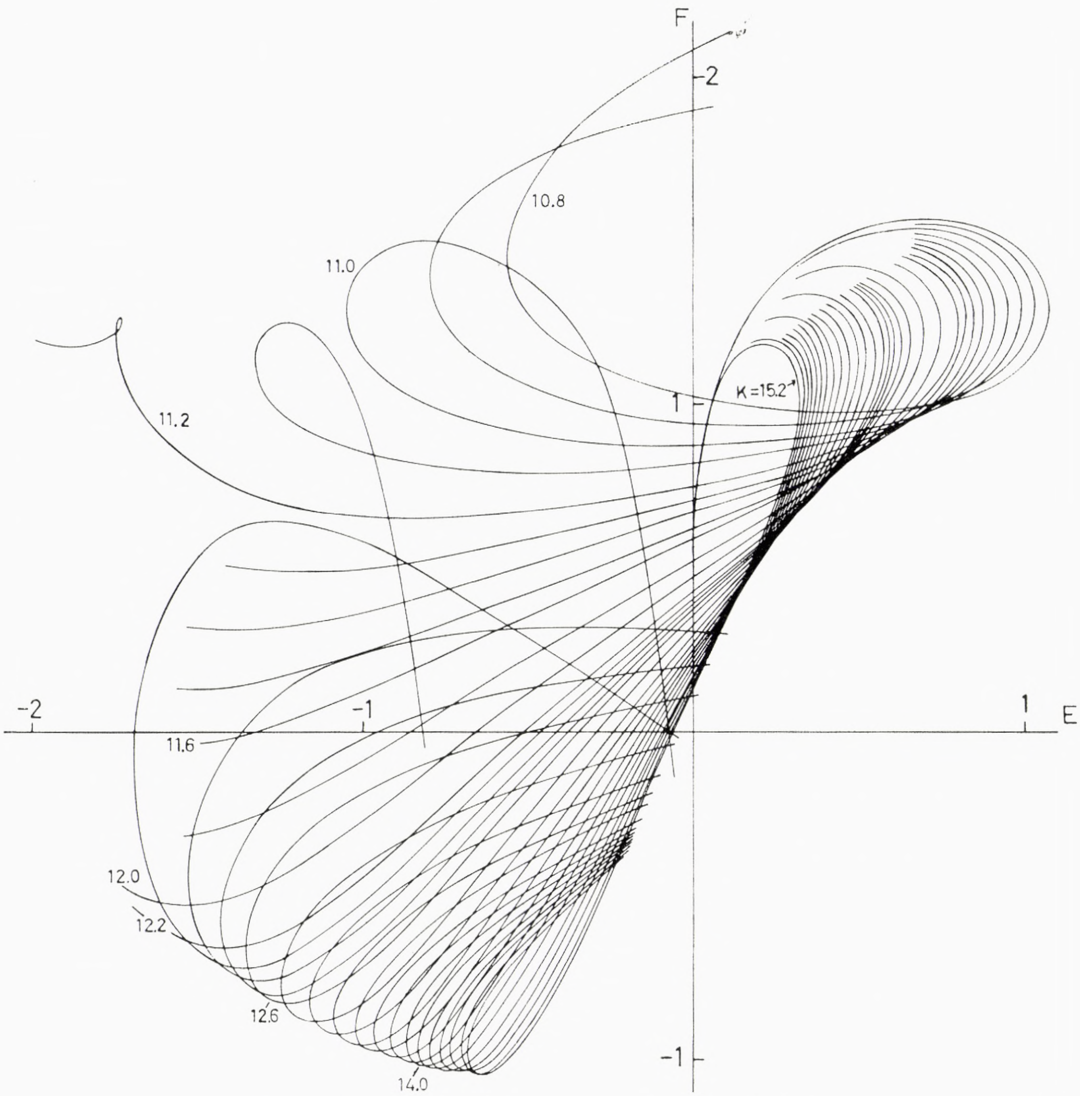

Figure 3 b: Ejection Orbits, $\dot{E}_{i} = 0$, $K = 10.8$ to 15.2.

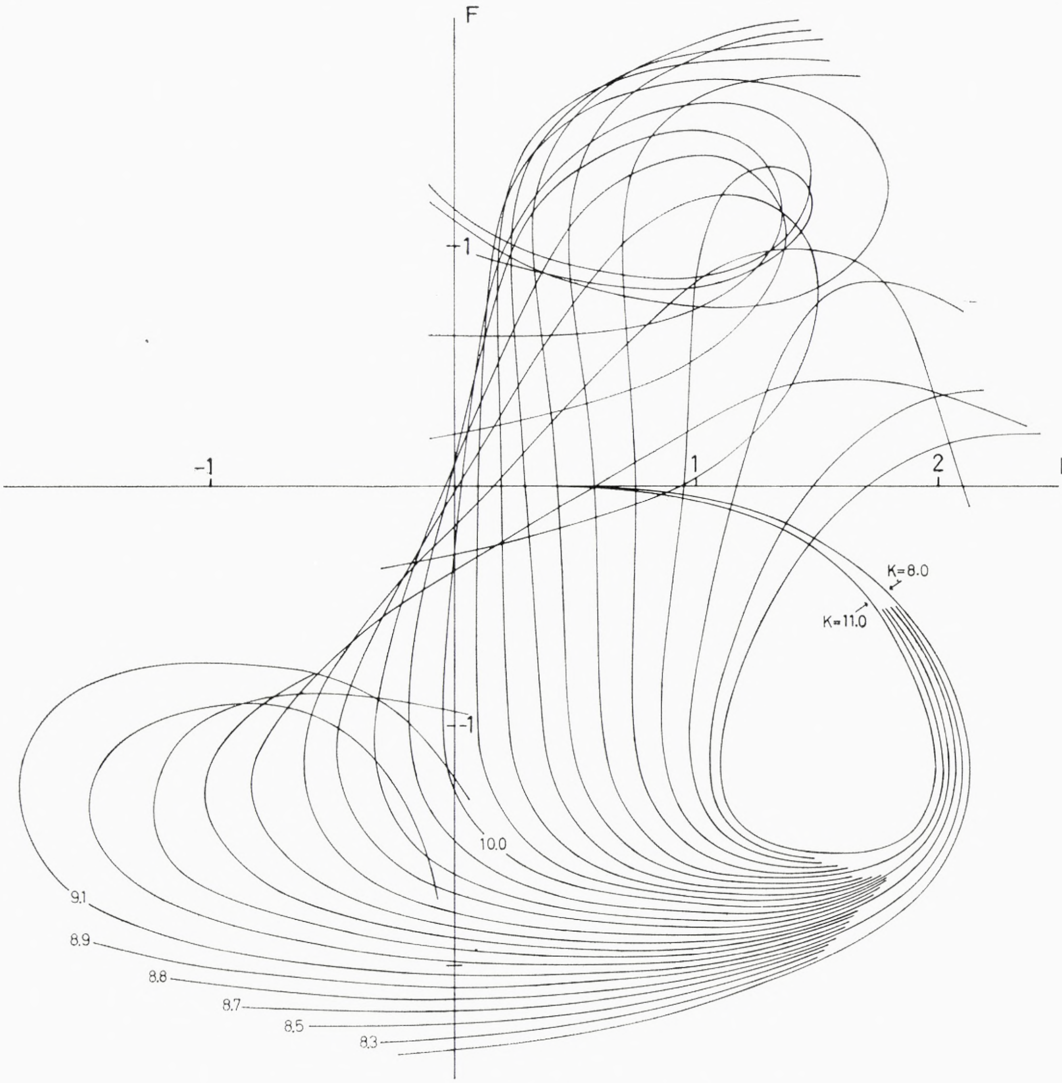

Figure 4 a: Ejection Orbits, $\dot{F}_i = 0$, $K = 8.0$ to 11.0.

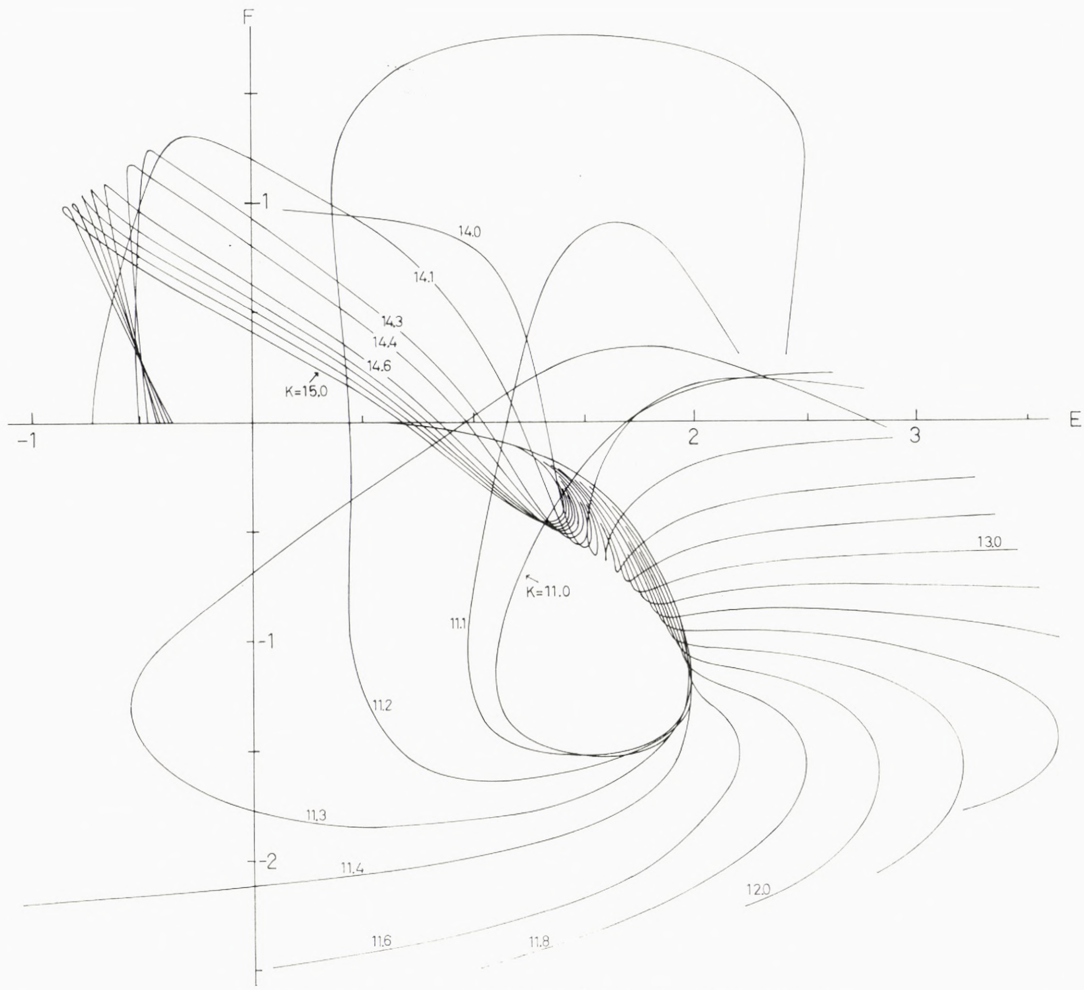

Figure 4 b: Ejection Orbits, $\dot{F}_i = 0$, $K = 11.0$ to 15.0.

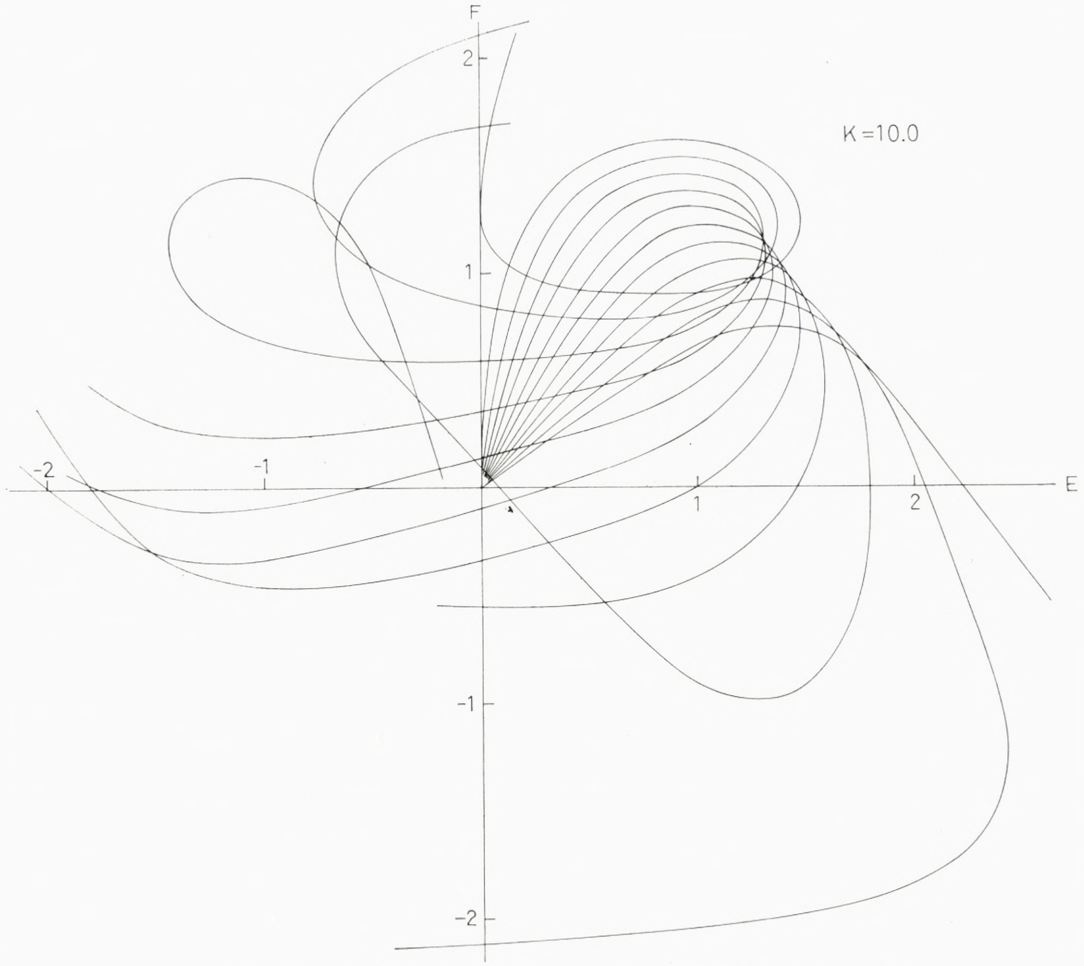

Figure 5 a: Ejection Orbits as a Function of Initial Angle measured from the + F-Axis, 0° to 60°, for $K = 10.0$.

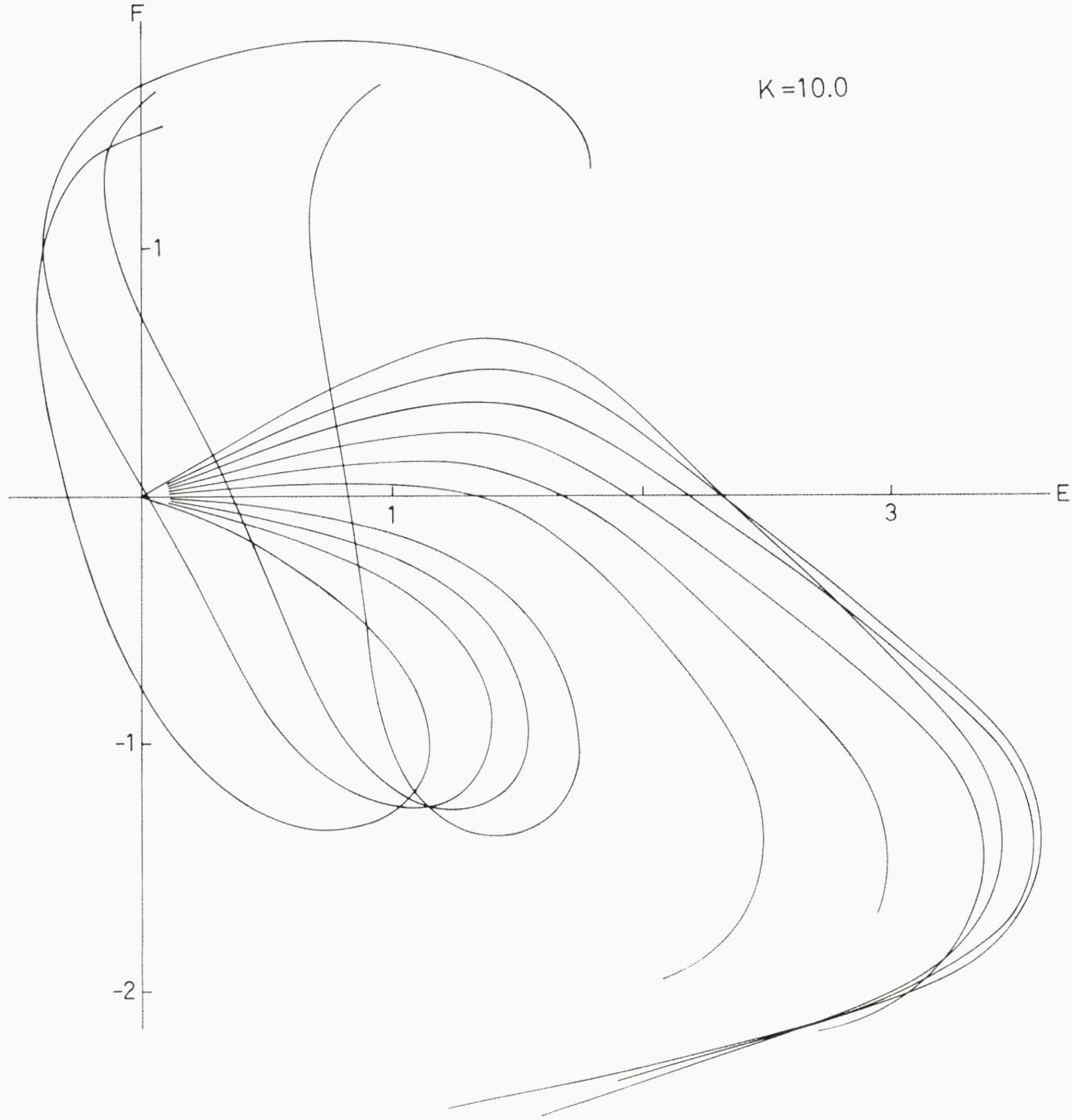

Figure 5 b: Ejection Orbits as a Function of Initial Angle measured from the + F-Axis, 60° to 120°, for $K = 10.0$.

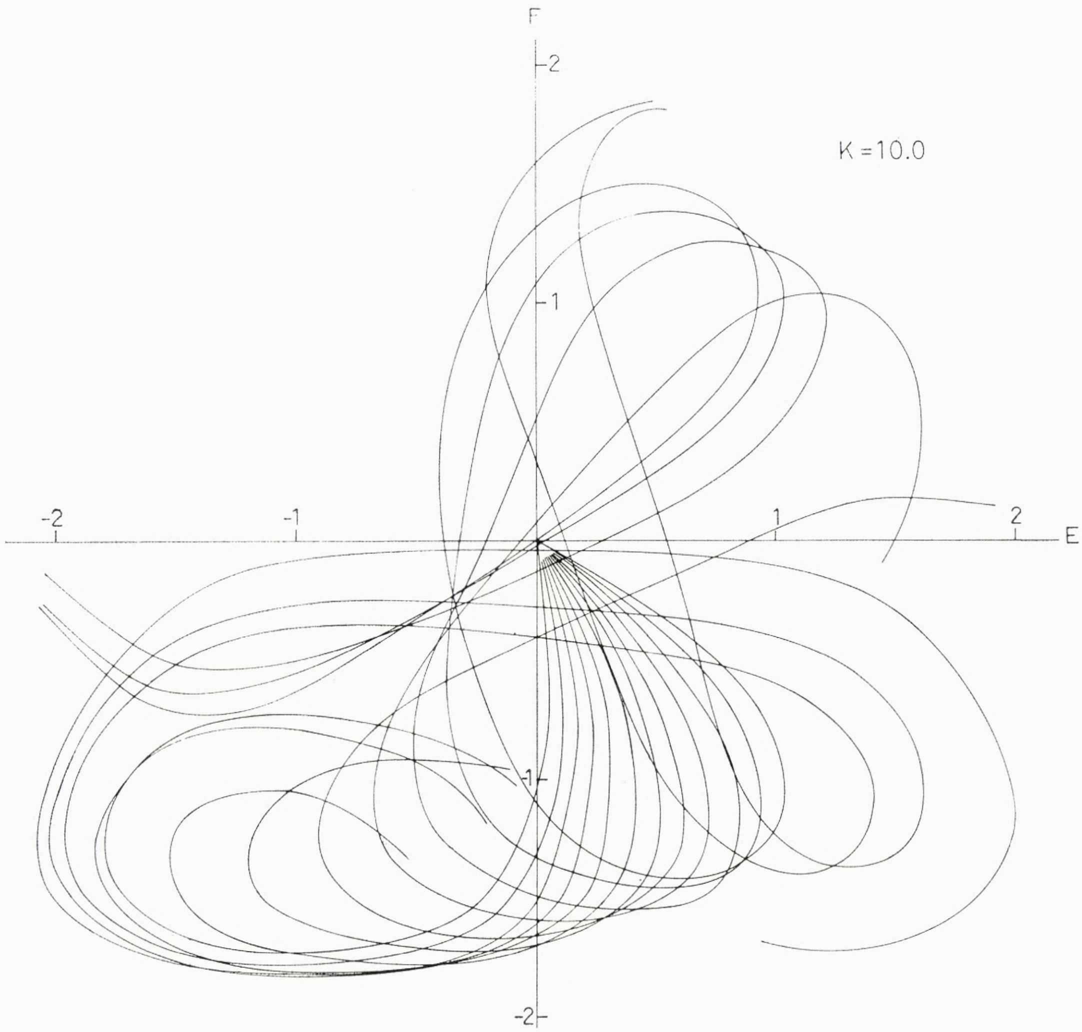

Figure 5 c: Ejection Orbits as a Function of Initial Angle measured from the + F-Axis, 120° to 180°, for K = 10.0.

Figure 6 a: Trajectories Normal to the F-Axis, $K = 12.5$, $F_i = -1.25$ to -0.50.
Over the latter part of its course, the trajectory for $F_i = -1.2$ follows rather closely that for $F_i = -1.15$.

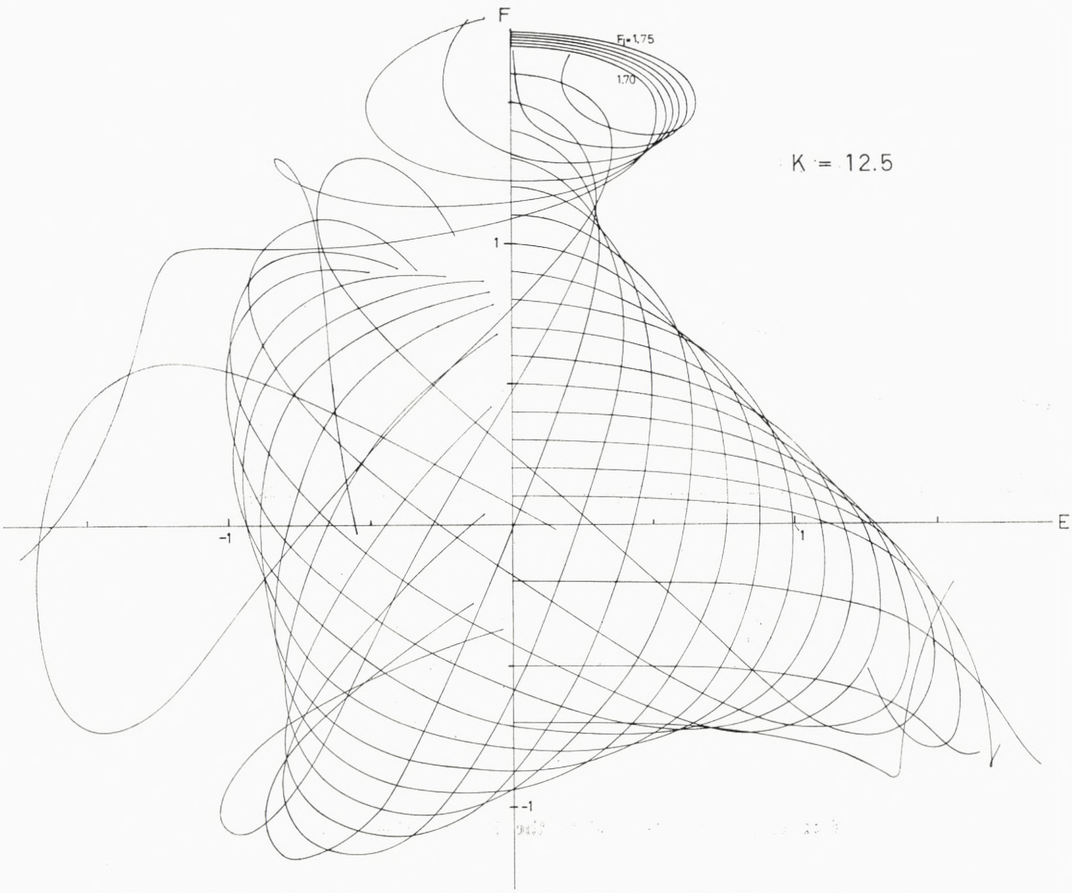

Figure 6 b: Trajectories Normal to the F-Axis, $K = 12.5$, $F_i = -0.70$ to 1.75.

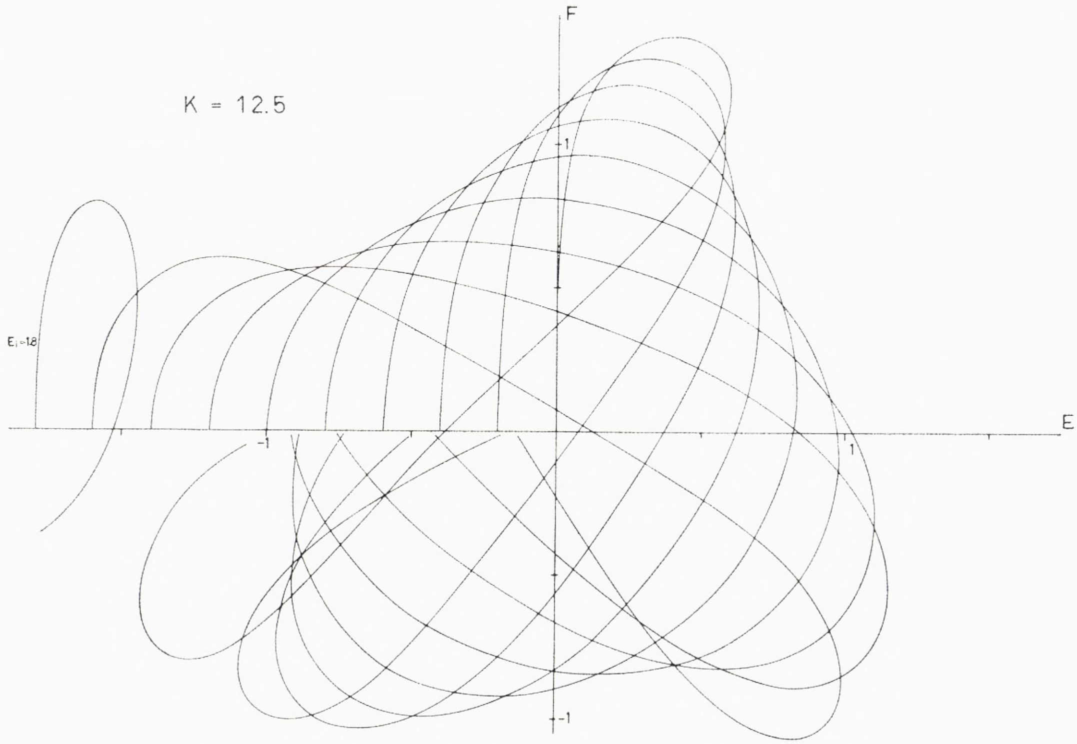

Figure 6 c: Trajectories Normal to the E-Axis, $K = 12.5$, $E_i = -1.80$ to 0.

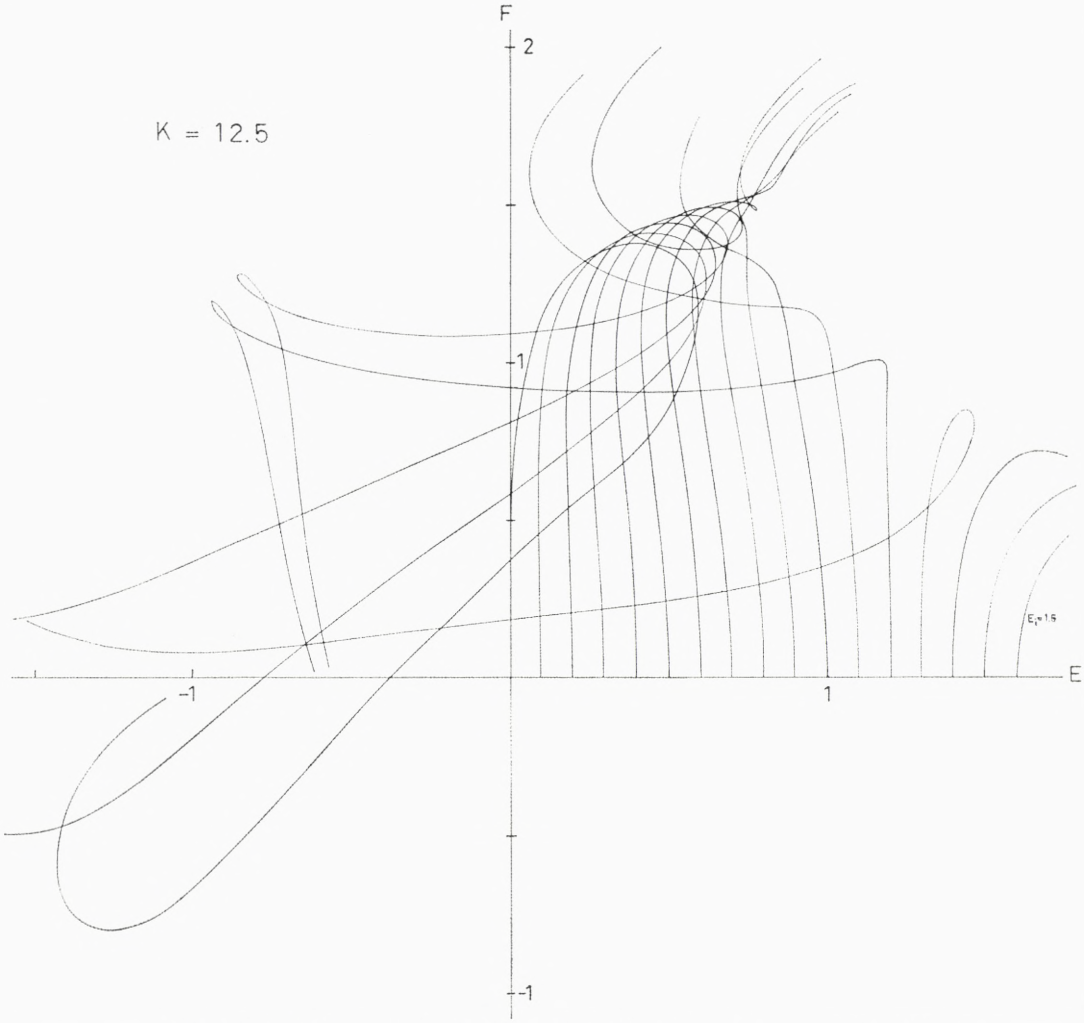

Figure 6 d: Trajectories Normal to the E-Axis, $K = 12.5$, $E_i = 0$ to $+1.60$.

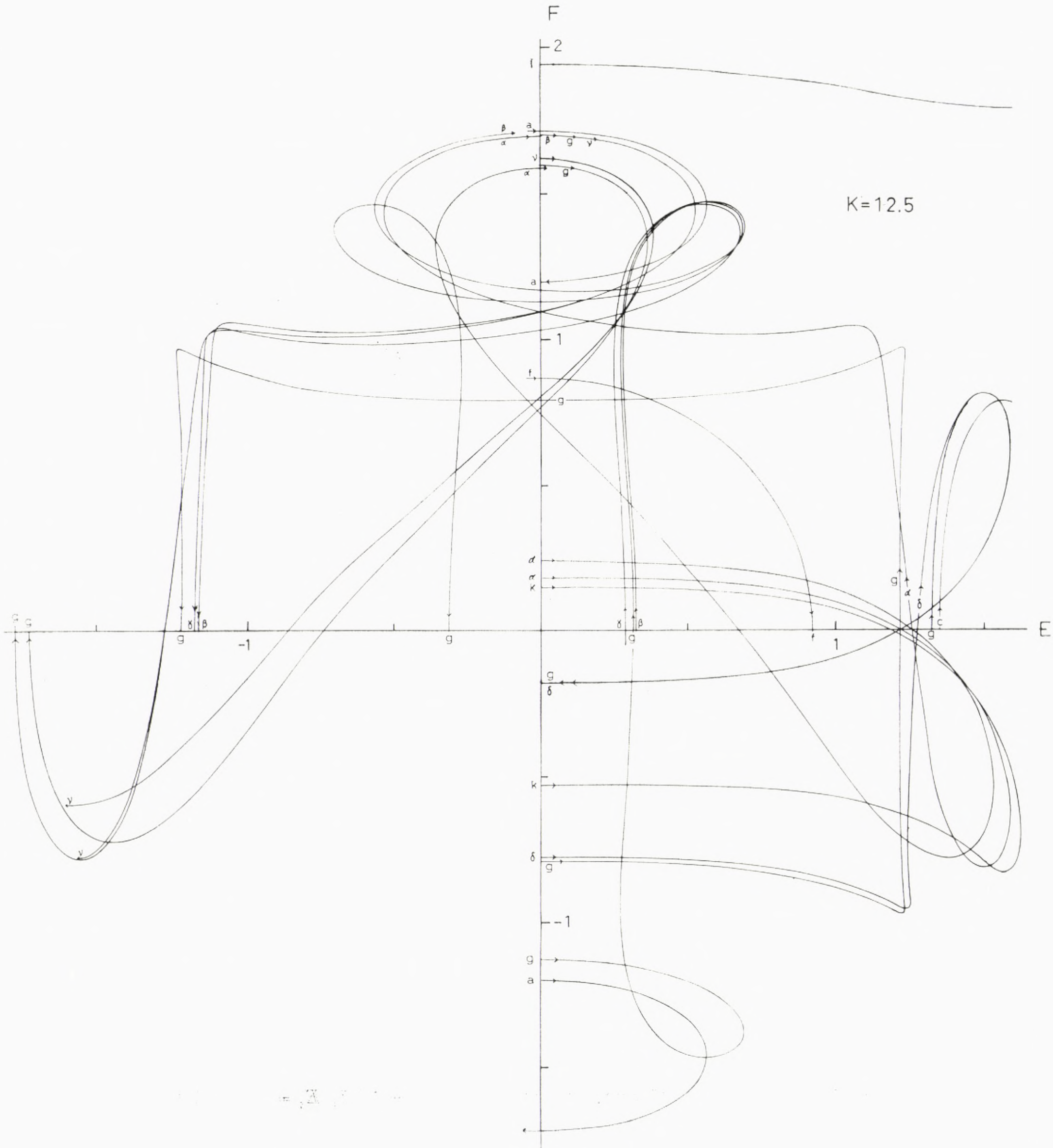

Figure 7: Simple Periodic Orbits for $K = 12.5$.

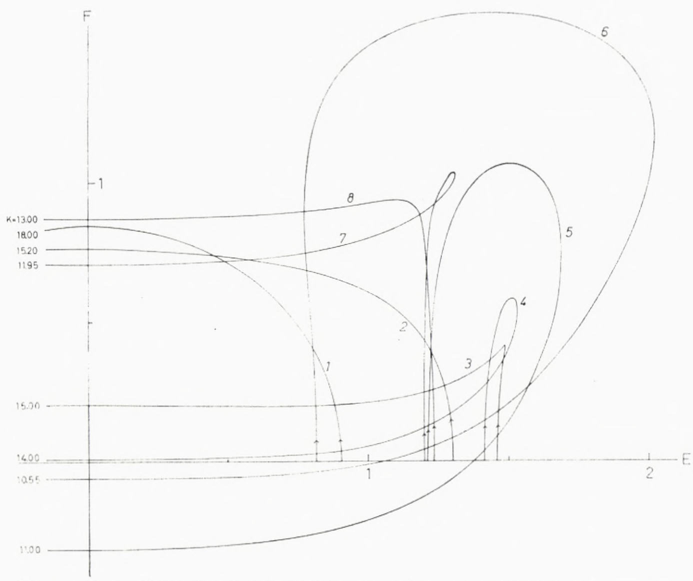

Figure 8 a: Development of the (g) Class, near mass m_2.
The curves are numbered in order along the profile (E or F) starting with 1 near the mass m_2 and increasing until the limiting orbit in Figure 8 c is reached.

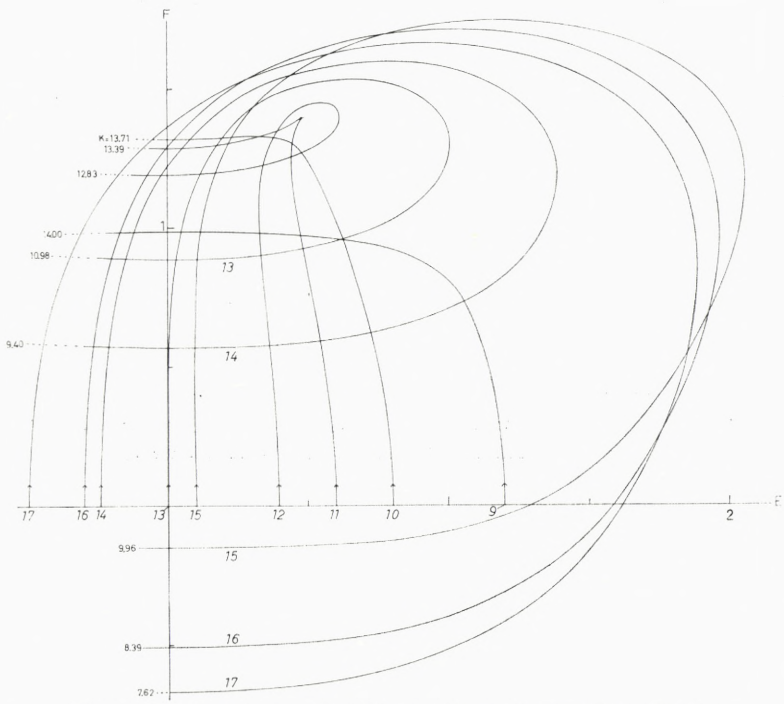

Figure 8 b: Development of the (g) Class, Intermediate Part.

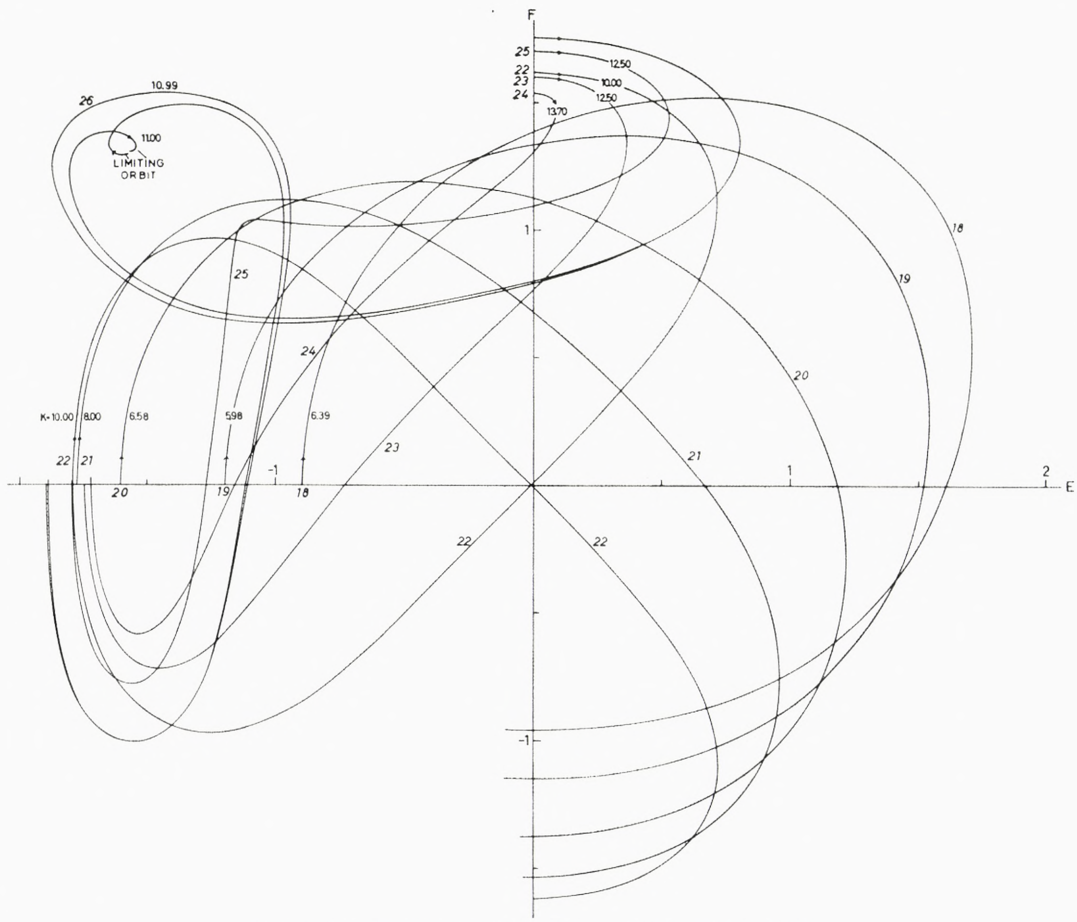

Figure 8 c: Development of the (g) Class, Termination.
The curves from 22 onwards are started on the F-axis, instead of on the E-axis, for clarity of representation and for comparison with Figure 7.

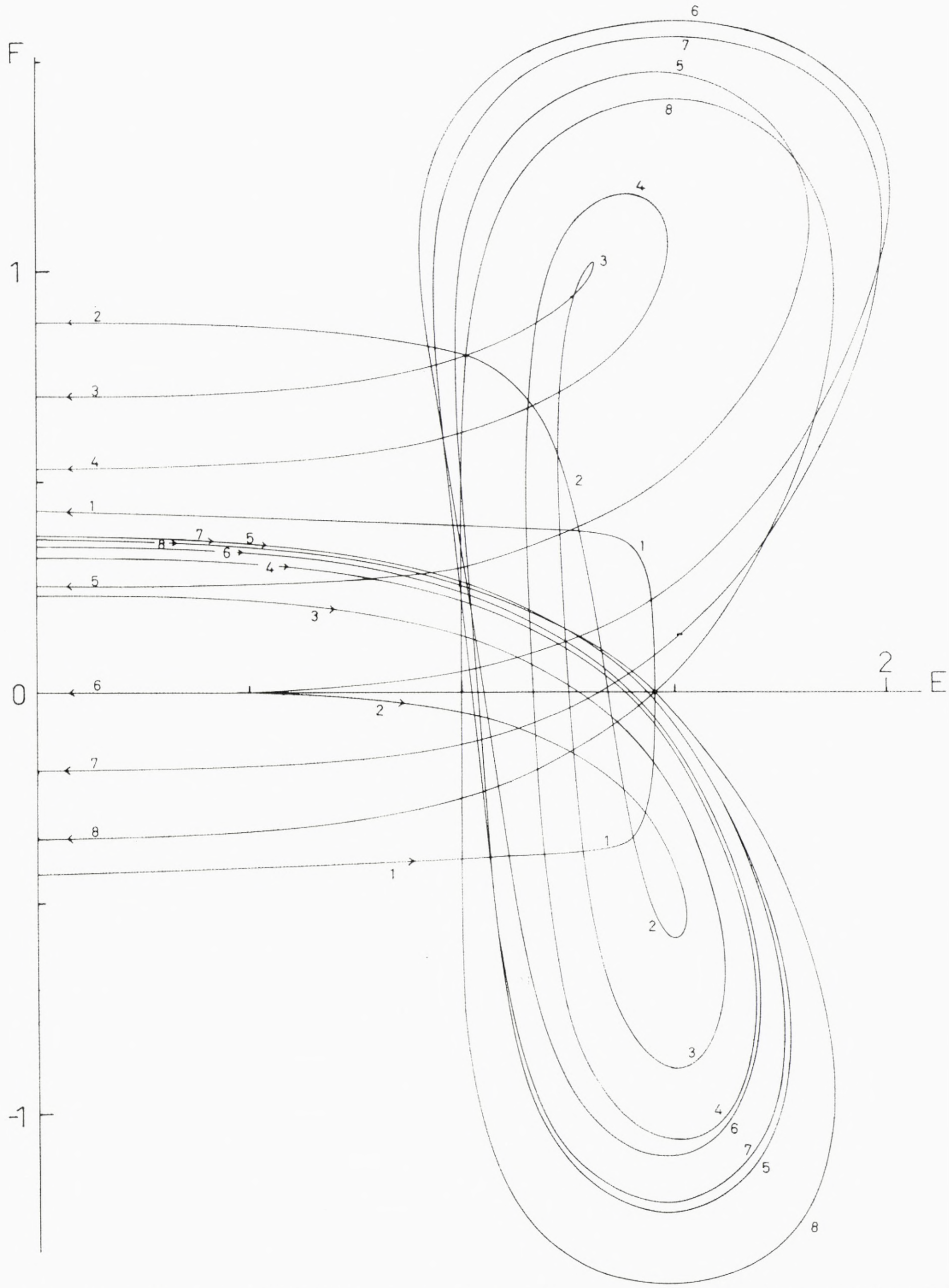

Figure 9: Development of the (δ) Class (a closed group), from maximum K to minimum K. (For the reverse development, take the mirror images about the E-axis).
The curves are numbered consecutively from 1 to 8, and have K-values of 15.428, 13.988, 12.0, 11.0, 10.23, 10.75, 10.30 and 9.915 respectively.

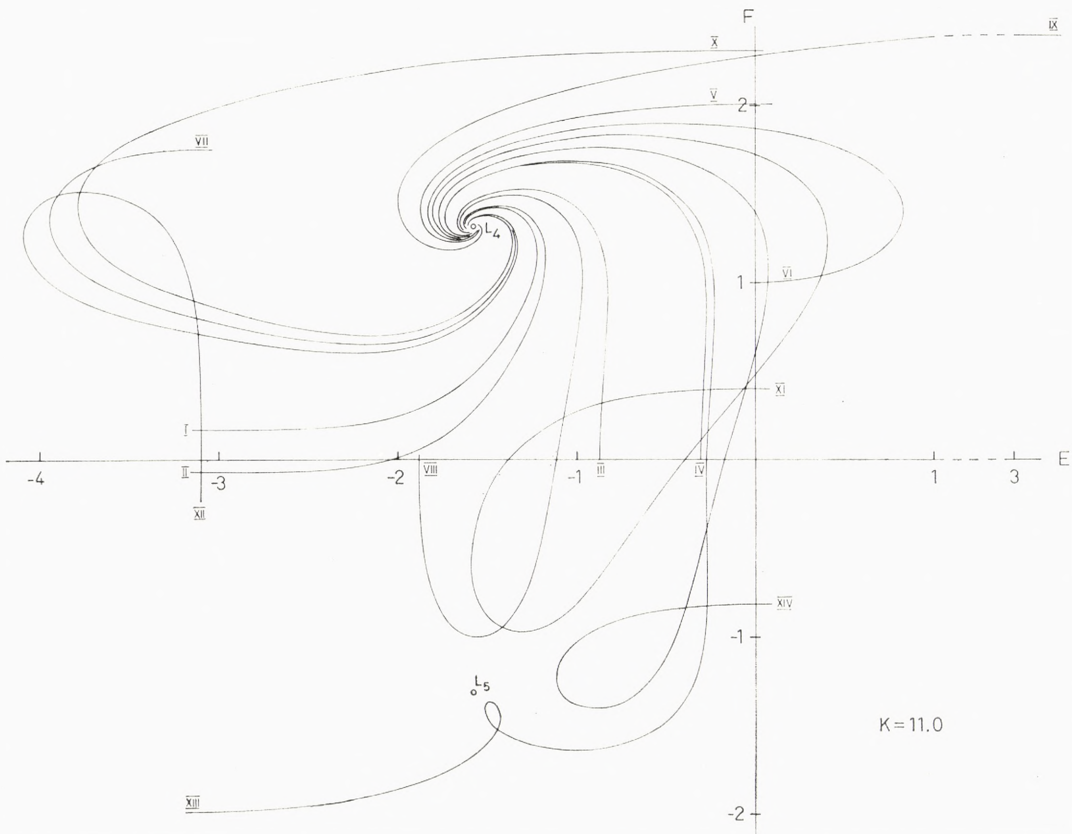

Figure 10: Periodic Limiting Orbits ($K = 11.0$), Symmetric with respect to the E- or F-Axis and Asymptotic to L_4 and L_5. (For complete half-orbits, take the proper mirror images to couple L_4 and L_5).
Curves VI, IX–XIV were calculated by C. Wagner.

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